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To the first of these is frequently applied the term "Constant of Precession." But as it is a variable quantity, the application of the term "constant" to it might be misleading. Indeed, no one of the four quantities just described can be regarded as a fundamental constant in astronomy, not only because they all vary, but because at least two of them are a result of both motions—that of the ecliptic and that of the equator.

In the case of the lunisolar precession, defined as the movement of the equinox upon a fixed ecliptic, there is some ambiguity. As the term is used by BESSEL and others, the fixed ecliptic is that of 1750, no matter for what epoch the precession may be defined. The annual lunisolar precession then diminishes with the time. But if we consider the precession upon the fixed ecliptic of the date, then the annual value is a quantity which increases with the time. When we have removed this ambiguity by adopting one or the other of these definitions, we may regard the lunisolar precession as dependent solely upon the motion of the equator, and as defining that motion. The same motion is defined by the quantity n, which expresses its annual value when measured on a great circle of the sphere. On the other hand, the two general precessions are resultants of both motions—that of the ecliptic and that of the equator.

To avoid the confusion which may thus arise in treating the subject, I have proposed the term "Precessional Constant" for a certain function P of the masses of the Sun, Earth, and Moon, and of the elements of the orbits of the Earth and Moon, on which the motion of the equator depends. This quantity changes only through the diminution of the Earth's eccentricity, and the change is so slight that it can become appreciable only after several centuries. From the value of P and the motion of the ecliptic, the four motions of the precession are derived by the formulæ,

$$
\begin{array}{ll}
\text{General precession;} & p = \mathrm{P} \cos \varepsilon - \varkappa \sin \mathrm{L} \cos \varepsilon \\
\text{Lunisolar precession;} & p_1 = \mathrm{P} \cos \varepsilon \\
\text{Precession in R. A.;} & m = \mathrm{P} \cos^2 \varepsilon - \varkappa \sin \mathrm{L} \operatorname{cosec} \varepsilon \\
\text{Precession in declination;} & n = \mathrm{P} \cos \varepsilon \sin \varepsilon
\end{array}
$$

Here L is the longitude of the instantaneous axis of rotation of the ecliptic, $\varkappa$ is its annual motion, and ε is its obliquity.

SECTION I.

Previous determinations.

Of the authoritative determinations of the precessional motion made during the nineteenth century the first is that of BESSEL. It rests upon a comparison of the star catalogue of the *Fundamenta Astronomiæ* with the catalogue of PIAZZI for 1800, the latter being first corrected by more recent Konigsberg observations. The values of the motions as they are quoted in ENGELMANN's edition of BESSEL's Abhandlungen from the Memoirs of the Berlin Academy of Sciences for 1815* are found below. But

*BESSEL's Abhandlungen, von R. Engelmann, Vol. I, pp. 262–285.

in the Tabulæ Regiomontanæ, published in 1830, somewhat different values are given. The two sets of values are as follows:

	Abhandlungen.	Tabulæ Regiomontanæ.
	$''$ $''$	$''$ $''$
Gen. prec.;	$50.18828 + .02442966T$	$50.22350 + .02443T$
Lunisolar;	$50.32832 - .02435890$	$50.36354 - .02436$
$m =$	$46.01135 + .03086450$	$46.04367 + .03086$
$n =$	$20.04554 - .009970204$	$20.05957 - .00970$

T is here counted from 1800.0. in terms of the century as the unit.

The interval between the two catalogues on which this determination rested was only forty-five years. In 1840, the shortness of this period and the fact that much material had accumulated for improving the values of the constants in question, led OTTO STRUVE to make a new determination. This determination, with some alterations by PETERS, has been most in use to the present time. It depends upon a comparison of the positions of 400 stars in the *Fundamenta* with determinations made principally at Dorpat about the mean epoch 1825. It therefore rests upon an interval of about seventy years.* The marked feature of this determination was that it included a determination of the solar motion, which was thus eliminated from the result. The corrections to the provisional value of the motion of precession for seventy years, which I suppose to be that of BESSEL, were found to be:

In R. A.; $\Delta m = +1''.6743$; $\Delta p = +1''.16 \pm 0''.67$
In Decl.; $\Delta n = +0''.2624$; $\Delta p = +0''.66 \pm 0''.86$
The combined result is $\Delta p = +0''.97 \pm 0''.53$.
$p = 50''.23492$ for the epoch 1790.

In the Astronomische Nachrichten, No. 485, STRUVE the elder quotes the result as

$$50''.23449 \pm 0''.00771$$

Neither of the STRUVES give any precessional motion but this. The motions with which STRUVE's name is associated are found in the memoir of PETERS, *Numerus Constans Nutationis*. Here, on page 160, PETERS gives $50''.3798$ as the lunisolar precession for 1800 found by STRUVE. He afterwards deduces the following values of the various quantities on which the precessional motions depend, as those resulting from STRUVE's work:

		$''$ $''$
General precession,	1800;	$50.2411 + .02268T$
Lunisolar precession,	1800;	$50.3798 - .02168$
	$m =$	$46.0623 + .02849$
	$n =$	$20.0607 - .00863$

These values are tabulated on page 200 of the paper, and are those in most general use up to the present time. It will, however, be seen that the value of the general

*Mémoires de l'Académie Impériale des Sciences de Saint-Pétersbourg, sixième série. Sciences Mathématiques et Physiques, Tome III.

precession is larger by 0″.43 per century than that found by STRUVE in his printed paper. I am not aware of the origin of this change, but its effects was to increase the error of a determination already somewhat too large.

As I pointed out to the Conference in 1896, some confusion has arisen from the fact that in applying PETERS's formulæ to the trigonometric reduction of star positions the obliquity of the equator to the fixed ecliptic has been carried only to terms in t^2, whereas, to make the formulæ consistent and homogeneous, this quantity should be carried to terms in t^3.

Before the above values had entirely superseded those of BESSEL, suspicion arose that they were too large, and that those of BESSEL were really nearer the truth, although the data on which they rested were much inferior. In his "Annales de l'Observatoire," Vol. II, LE VERRIER adopts the following values:

	″	″
General precession (1850);	50.23572	$+ .022578T$
Lunisolar precession;	50.37140	$- .021762$
	$m = 46.06010$	$+ .028373$
	$n = 20.05240$	$- .008663$

These have been used in the reductions of stars at the Paris Observatory.

In his "*Lehrbuch zur Bahnbestimmung der Kometen und Planeten,*" Vol. I second edition, OPPOLZER develops the most exhaustive theory of precession which has yet appeared, but without any discussion of the observations on which his adopted values of the constants depend. His results are (pp. 202, 206):

	″	″
General precession (1850);	$50\ 23465$	$+ .022580T$
Lunisolar precession;	50.36924	$- .021776$
	$m = 46.05931$	$+ .028390$
	$n = 20.05150$	$- .008668$

About 1886 LUDWIG STRUVE took advantage of the completion of AUWERS' reduction of BRADLEY's observations, and of the POULKOWA catalogue for 1845, 1855, and 1865, to enter upon the most thorough discussion of the observations which has yet been published. The negative correction found to OTTO STRUVE's result was quite large. The precessional motions m and n, found by L. STRUVE, were

	″	″
General precession (1850);	50.2283	$+ .02120T$
Lunisolar precession;	50.3407	$- .02132$
	$m = 46.0554$	$+ .02741$
	$n = 20.0452$	$- .00849$

It must be stated, however, that the smallness of these values arises partly from the large positive correction which AUWERS applied to BRADLEY's Right Ascensions.

When the various determinations are reduced to a common equinox, it is found that the discordances are materially diminished.*

All the preceding determinations rest upon BRADLEY stars, which I conceive to afford the best material available at the present time. Determinations from a great number of faint stars have been made by DREYER, NYREN, and BOLTE.

DREYER'S investigation is found in the Journal *Copernicus*, Volume II, Dublin, 1882. It is prefaced by a very exhaustive history of previous determinations of this constant. The material consists of LALANDE'S catalogue, and the catalogue of SCHJELLERUP for 1865. The system of right ascensions to which both catalogues are reduced is N_1. The results for $1800 + t$ are—

$$\begin{aligned} \text{Lunisolar precession} &= 50''.3752 - 0''.0002168t \\ \text{General precession} &= 50.2365 + 0.0002268t \\ m &= 46.0581 + 0.0002849t \\ n &= 20.0589 - 0.0000862t \end{aligned}$$

NYRÉN'S result was based on a comparison of stars in BESSEL zones with SCHJELLERUP, and is much smaller than the above. The motion is, however, greatly increased when the two catalogues are reduced to the system N_1, so that, ultimately, the discordance is not great. But the brevity of the interval renders the determination weak.

BOLTE'S investigation is based on the declinations of LALANDE as compared with SCHJELLERUP. His work was published only as a doctorial thesis, and is therefore not so well known as it should be.

These determinations would be better than any that could be made from BRADLEY stars were it not for the probable systematic errors in the observations and reductions. First among these errors is that in Right Ascension depending on the magnitude of the star. When we come down to faint stars this error may increase to an almost unknown extent. Moreover, as a general remark applicable to all zone observations, it may be said that there is great room for systematic errors, both in the Right Ascension and Declination, which would materially affect the value of the precessional motion derived from them.

List of the principal determinations.

For the sake of easy comparison the values above given, with some others, are reduced to 1850, and the results shown below.

In the reductions of the earlier values to 1850 the motion in my "Elements and Constants" has been used.

The value N_0 is the preliminary one, used in the tables of the planets and in my "Elements and Constants."

*See Vierteljahrsschrift der Astronomischen Gesellschaft, Vol. XIII, 1878, page 107.

The value N_{96} is that reached in the present paper.

	General precession.	m.	n.
	"	"	"
Bessel I,	50.1994	46.0253	20.0413
Bessel II,	50.2346	46.0576	20.0553
Struve,	50.2478		
Peters,	50.2522	46.0763	20.0564
Leverrier,	50.2357	46.0601	20.0524
Oppolzer,	50.2346	46.0593	20.0515
L. Struve,	50.2283	46.0554	20.0452
N_0,	50.2371	46.0636	20.0479
Dreyer,	50.2478	46.0723	20.0546
Nyrén,	50.1995	46.0426	
Bolte,			20.0537
N_{96},	50.2453	46.0711	20.0511

Section II.

Material used in the present discussion.

In starting with a new determination, the first question to be considered is what material we shall use. What we want is the annual motion of the equinox, arising from the combined motions of the equator and the ecliptic, relative to an axis absolutely fixed in space. As observations can not be referred to any line or plane which we know to be absolutely fixed, we are obliged to assume the general mean position of the fixed stars as affording the invariable axes of reference. Here we meet with difficulties arising from the proper motions of the stars, and especially from the solar motion. The elimination of the latter from the mean result is a matter of especial difficulty.

It is also to be recalled that the position of the equinox relative to the mean stellar sphere, assumed as fixed, requires three practically separate determinations. These are—

1. The correction to the Sun's absolute mean right ascension or longitude. This is obtained principally from observations of the Sun's declination.

2. The correction to the general mean right ascension of the clock or standard stars relative to the Sun.

3. The determination of the positions of the clock stars relative to the great mass of stars assumed to define the fixed celestial sphere.

In the present investigation it is needless to consider the first two determinations because, by a resolution of the Conference of 1896, in accordance with which this investigation is made, the position of the equinox among the clock stars is defined by the system N_1, formed by those equatorial stars whose position is given in the Appendix to the *Catalogue of* 1098 *Clock and Zodiacal Stars*, found in Volume I of these *Astronomical Papers*. It is therefore only necessary, so far as the Right Ascensions are concerned, to determine the positions of the great mass of stars relative to the standard stars in question.

The question now arises, what stars we shall choose. Undoubtedly, the great mass of the fainter stars, down to the ninth magnitude, which we find in various zones observed during the present century, would be preferable. But a determination made from these stars would be subject to very large systematic error, arising from the shortness of the interval between the earlier and the later observations. This uncertainty is magnified by the possibility of very large personal error due to the small magnitude of the star. It is now well known that, at the present time, especially when observations are made by the chronograph, faint stars are observed too late relatively to brighter ones. Were this personal error constant at all times and for all observers during the century, no error would thus arise. But there is strong reason for suspecting that this error not only varies from one observer to another, but that it is much larger for the recent than for the older observations. If such is the case, the precessional motion derived from such a comparison would be too great. The probability of this error has led me to depend mainly upon observations of the brighter stars, especially those in BRADLEY's catalogue. As the total number of stars exceeds 3,000, it may be expected that all accidental errors will be nearly eliminated from the mean. Though the systematic error just considered may not be wholly eliminated, it will certainly be much smaller than in the case of a determination founded on much fainter stars observed during a shorter interval.

I have, however, made an exception of the LALANDE catalogue by using the comparisons of this catalogue with those of BOSS and SCHJELLERUP. The former determined his personal error dependent on magnitude. In the case of SCHJELLERUP this error seems to be small.

SECTION III.

Provisional precessions.

From the considerations already set forth, the desirableness will be seen of adopting at the start a provisional value of the precessional constant which will be as near as practicable to the truth. That STRUVE's value needs a large negative correction is now widely admitted, and, it seems to the writer, made quite clear by the data already set forth in the preceding pages. I shall, therefore, take as provisional values the precessional motion already alluded to, which is employed in my "Elements and Constants," as the provisional value to be corrected.

The principal quantities for the epoch 1850 are these:

The precessional constant for a solar century of 36524.2 days

$$P = 5489''.78 - 0''.00364T$$

$$\begin{aligned} &\qquad\qquad\qquad\quad '' \\ \text{General precession; } p &= 5023.71 \\ \text{Lunisolar precession; } p_1 &= 5036.02 \\ 100\,m &= 4606.36 \\ 100\,n &= 2004.79 \end{aligned}$$

The fundamental quantity to be determined is the correction to this value of P; but it will be more convenient to take p_1, the lunisolar precession, as the quantity to be

corrected. The correction I shall call δp, or simple p, so that the true lunisolar precession sought shall be

$$p_1 = 5036''.02 + \delta p, \text{ or } 5036''.02 + p$$

SECTION IV.

Systematic corrections to the proper motions.

I have taken, as the basis of the work, AUWERS' proper motions of the BRADLEY Stars, as found in Vol. III, Abth. II of his reduction of BRADLEY'S observations. But it is necessary to correct the proper motions of AUWERS so as to reduce them to the system already employed for the fundamental stars. These motions, it will be recalled, are derived by a comparison of the BRADLEY catalogue for 1755 with a number of the best recent determinations. The latter were reduced to a mean of the Greenwich catalogues for 1860 and 1864. The difference between the mean corrections of these catalogues and of the BRADLEY catalogue will therefore be the correction to the proper motions for about one hundred and five years.

In the Declinations, however, we have a simpler way of proceeding. The proper motions in question were incorporated in the A. G. C. fundamental catalogue, Publication XIV of the *Astronomische Gesellschaft*, and may therefore be considered as systematically identical with these. Now, in the work on the fundamental stars, I have determined the correction to reduce the declinations of the A. G. C. to those of the normal system, together with their centennial variations. The latter are clearly the corrections to be applied to the proper motions of AUWERS-BRADLEY in Declination.

The systematic corrections to the proper motions in question thus consist of two classes; those for reduction to the normal system, and those for reduction to the new provisional value N_0 of the precessional constant.

It is found that, for reduction to the system N_1, the Right Ascensions of BRADLEY required the constant correction for equinox

$$\Delta\alpha\ 1755 = -0^s.079$$

The modern standards to which the Right Ascensions are reduced in publication XIV, and in AUWERS-BRADLEY, is that of the two Greenwich seven-year catalogues for 1860 and 1864, the Right Ascensions of the second one, that for 1864, being first increased by $0^s.01$. By the comparison of these catalogues with the system N_1 it appears that this system would require the following constant correction for equinox:

$$\text{Catalogue for 1860 } N_1 - A = N_1 - \text{Grh.} \qquad = +0^s.024$$
$$\text{Catalogue for 1864 } N_1 - A = N_1 - \text{Grh.} - 0^s.01 = +0^s.024$$

There is, therefore, an exact agreement between the results derived by comparison with the two catalogues. Thus the constant corrections to the A. G. system on account of equinox are

	s.
Epoch 1755	−.079
Epoch 1860	+.024
Correction for 105 years, . .	+.103
Correction for 100 years, . .	+.098

a result which agrees exactly with that found by direct comparison of N_1 with A. G. C.

We must also apply the mean periodic term for the two Greenwich Catalogues, of which the value is

$$\Delta\alpha = +{}^{s}.014 \cos\alpha - {}^{s}.011 \sin\alpha = {}^{s}.018 \cos(\alpha + 38^\circ)$$

If to the constant ${}^{s}.098$ we apply the difference of the corrections $\Delta\alpha_\delta$ for the two catalogues, we shall have the following values of the systematic corrections to AUWERS' proper motions in Right Ascension in order to reduce them to the modern system and to STRUVE'S constant.

Dec.	Corr.
°	s.
−30	+.140
−20	.129
−10	.126
−0	.114
+10	.097
20	.082
30	.075
40	.102
50	.096
60	.14
70	.16
80	.16
90	.16

For the precessions at the mid-epoch 1810, AUWERS uses

	$''$	$''$
$m =$	4606.51;	$n =$ 2005.98
N_0 has	4605.24	2005.13
$A - N_0 =$	+1.27	+0.85

Hence, the centennial proper motions of AUWERS in Right Ascension require the correction

$$+1''.27 + 0''.85 \sin\alpha \tan\delta$$
$$= 0^{s}.085 + 0^{s}.057 \sin\alpha \tan\delta$$

In the equations of condition the proper motions are multiplied by cosine δ. If we put C for the numbers given above as a function of the Declination increased by $1''.27$, the complete reduction required is given by the expression

$$\Delta\alpha \cos\delta = C \cos\delta + 0''.27 \cos\delta \cos(\alpha + 38^\circ) + 0''.85 \sin\alpha \sin\delta$$

This quantity has been tabulated in a table of double entry as a function of both coordinates.

In the case of the Declinations the corresponding corrections to the proper motions will consist of the following three parts:

1. The coefficient of T in the reduction of the A. G. to the new normal system C_{96}, which correction is a function of the Declination alone.

2. The negative of the correction $\Delta\delta_a$ to AUWERS-BRADLEY, which will be tabulated subsequently in Part II as a function of the Right Ascension and Declination.

3. The reduction for the precession N_0, of which the value is $\Delta\mu' = +0''.85 \cos\alpha$.

The sum of these three quantities also was tabulated in a table of double entry.

The tables of the preceding corrections constructed for use in the work are not printed in the present paper.

SECTION V.

Formation and treatment of the equations of condition.

The fundamental quantity of which the correction is to be found is that already defined as the precessional constant. But we may equally take for correction the value of the lunisolar precession

$$p_1 = P \cos\varepsilon$$

as the quantity for which the correction is to be determined, because the value of the obliquity ε is so well known that the correction of p_1, may be regarded as a constant. We shall then have

$$m = p_1 \cos\varepsilon - \varkappa \sin L \operatorname{cosec}\varepsilon$$
$$n = p_1 \sin\varepsilon$$

The planetary precession, $\varkappa \sin L \operatorname{cosec}\varepsilon$, can be derived only from theory, and the value already found from the masses of the planets must be adhered to. We therefore have for the symbolic corrections to m and n

$$\delta\, m = \cos\varepsilon\ \delta\, p$$
$$\delta\, n = \sin\varepsilon\ \delta\, p$$

The annual precession in R. A. being

$$m + n \sin\alpha \tan\delta$$

its symbolic correction is

$$\delta\, p\, (\cos\varepsilon + \sin\varepsilon \sin\alpha \tan\delta)$$

If we put μ and μ' for the proper motions in Right Ascension and Declination as derived with the provisional value of the precession, the equations of correction will be those which will reduce this proper motion to zero. Omitting for the moment the solar motion, these equations will be

$$\delta\, p\, (\cos\varepsilon \cos\delta + \sin\varepsilon \sin\alpha \sin\delta) = \mu \cos\delta$$
$$\delta\, p \sin\varepsilon \cos\alpha = \mu'$$

If we add to our equations the terms depending on the coordinates X, Y, Z of solar motion, the complete equations of condition will be

$$a\,\delta\, p + \frac{\sin\alpha}{\rho} X - \frac{\cos\alpha}{\rho} Y = \mu \cos\delta \qquad (1)$$

$$b\,\delta\, p + \frac{\cos\alpha \sin\delta}{\rho} X + \frac{\sin\alpha \sin\delta}{\rho} Y - \frac{\cos\delta}{\rho} Z = \mu'$$

where we have put

$$a = \cos \varepsilon \cos \delta + \sin \varepsilon \sin \delta \sin \alpha$$
$$b = \sin \varepsilon \cos \alpha$$

These equations are well-known ones heretofore used by others. I have omitted all expressions representing a possible revolution of the stellar system, believing that their introduction would be useless.

We shall have as many pairs of equations of this kind as there are stars whose proper motions are used. From these equations we are to derive the values of the three components of solar motion X, Y, and Z, and the correction of the lunisolar precession δp. But in proceeding to subject these equations to the method of least squares, we meet with two serious and embarrassing difficulties. The first of these arises from the presence in each equation of the quantity ρ, the distance of the star. We have no criterion for the value of ρ except the proper motion of the star and its magnitude. Both of these criteria are extremely uncertain, owing to the great diversity in the actual magnitudes of the different stars and the actual amount of their proper motions.

A very serious objection to using the proper motion of a star as a criterion of its distance is that we thereby assume the solution of the very problem at which we are aiming. The amount of the computed proper motion is a function of the assumed value of the precessional constant, and is also affected by the solar motion, which is inseparably connected with the determination of the precessional constant. If, then, we select those stars which have very small apparent proper motions, our selection will not consist of those whose motions are absolutely the smallest, but of those which give small corrections to the precessional constant and to the adopted value of the solar motion. Thus will arise a bias in the direction of the values with which we start. I therefore conclude that, while it will be advisable ultimately to assign weights dependent on the amount of the proper motion, we can do this only with the residual proper motions which remain after making the best allowance we can for the solar motion and for the correction to the precessional constant.

This reasoning applies mainly to the cases of individual stars. If we consider large classes of stars, the force of the objection will be greatly diminished, since the mean amount of the proper motion will probably then approximate to the inverse of the mean distance. We shall therefore follow previous investigators in classifying the stars according to their magnitudes. We shall then estimate the mean distance, or, rather, the mean parallax, of the stars of each class to be proportional to the mean value of the proper motion for the individual stars of the class. This proceeding will have the advantage of enabling us to make a combination by which the uncertainty as to the mean value of ρ in each case will be merged with the question of the accidental errors, as follows:

Let us multiply the preceding equations by ρ. The equations of condition (1) will then take the following form:

$$\left.\begin{array}{l} a \rho\, \delta p + \sin \alpha\, X - \cos \alpha\, Y = \mu\, \rho \cos \delta = \rho n \\ b \rho\, \delta p + \cos \alpha \sin \delta\, X + \sin \alpha \sin \delta\, Y - \cos \delta\, Z = \mu' \rho = \rho n' \end{array}\right\} \quad (2)$$

It is quite indifferent in writing the equations whether we consider the latter in this or their original form, since we shall have in either case to multiply each equation

by such factor or weight as will reduce the probable errors of the second term to a constant value for all the equations. If in using the first or original form we have the system of weights

$$w_1;\ w_2;\ w_3;\ \text{etc.},$$

it is evident that in using the second form our system of weights should be

$$\frac{w_1}{\rho_1^2};\ \frac{w_2}{\rho_2^2};\ \frac{w_3}{\rho_3^2};\ \text{etc.}$$

We shall then have the same equation, whether we start with one form or the other In either case the weight to be chosen will be that which will reduce the probable error of the absolute terms to the same value in the case of each group of stars. This probable error is composed of two parts—the absolute proper motions of the stars themselves, as seen from the earth, and the errors of their determinations. But it is unnecessary to consider these two sources of error separately, because their combined effect is shown by the mean divergence of the individual stars from the general mean in the case of each class of stars.

It follows that if we make the proposed classification, and find the mean value of the residual proper motion in the case of the stars of each class, the square roots of these means will form a series of divisors for the several equations of condition which will reduce the probable errors of all to the same value.

This difficulty disposed of, there remains one yet more embarrassing, arising from the fact that the proper motions, which we are compelled to treat as residual errors, do not follow the normal law of error, even when reduced to one standard of mean parallax in the way proposed

In all determinations of the precessional motion heretofore made, the proper motions of the stars have been treated as residual errors by the application of the method of least squares. The acceptance of a mean of several different observed values of the same quantity as the most likely value is a special example of this method. But the method has to be modified, in so far that stars of excessively large proper motion are arbitrarily thrown out. LUDWIG STRUVE, in his discussion on the subject, adopted the principle of throwing out those stars whose proper motion was more than ten times the mean proper motion of stars having that magnitude. But it seems that this led to the rejection of only seven stars in all.

That this system of rejecting all results which diverge largely from the general mean is logically sound has long been understood. But there is an evident want of system in the proceeding, because we assume a certain fixed limit within which results are accepted with their full weight, while without the limit they are totally rejected. The quantities which deviate most largely are those whose rejection or retention will most influence the concluded result. Consequently, the final result will frequently depend largely on where we set the limit of rejection, and this is a matter which has to be left entirely to the judgment of the investigator. I have on several occasions pointed out that the cases in which the method of least squares gives the best result are

not the typical ones of astronomy.* The general propositions which relate to the subject may be formulated as follows:

I. The method of least squares gives the best value when and only when the probability of making an error of a given amount x is expressible as a function of x by the formulæ,

$$dp = \frac{h}{\sqrt{\pi}} e^{-hhxx} dx$$

II. In astronomical practice it commonly happens that the actual probability of an error of given magnitude x is not given by a function of this form, but deviates from this function in such a way that, when the closest adjustment to the formula is made, small errors and large errors are more frequent than intermediate ones.

III. In this case the best result is obtained by giving to the more divergent results a weight depending on the magnitude of the divergence, gradually converging toward zero as the divergence increases.

IV. The necessity of thus modifying the method of least squares becomes greater the wider the deviation of the actual distribution of error from that given by the exponential formulæ.

In order to judge of the best mode of procedure, I begin by arranging the reduced proper motions of the Bradley stars in the direction of R. A.—that is, the values of $\mu \cos \delta$ derived by the methods already described, in such a way as to show their distribution statistically. In order to separate the effect of the parallactic motion, I have made the classification not only according to magnitude, but according to the quadrant of right ascension. The adopted magnitudes are those given by Auwers in the catalogue. As the stars of the first magnitude are too few to be used by themselves, I have put all down to magnitude 2.9 in one group. A second group includes those from 3.0 to 3.9; a third those of each following magnitude down to 6.9. All below this magnitude I have included in one group as of magnitude 7. For each quadrant of right ascension and each group I have taken the proper motions in right ascension and made a count of the number having certain values. One of the subgroups thus formed comprises the proper motions which come out as exactly zero. The negative proper motions are then classified as $-0''$. and a fraction, $-1''$. and a fraction, etc., up to $-16''$. and a fraction. All of $17''.0$ and higher are put into one group. A similar division is made on the positive side. The result of this process is shown in Table I.

Whatever law of probability of error we adopt, there are always two quantities expressive of the mean magnitude of the error which admit of rigorous definition. One of these is the mean, the other the probable error. The mean of the squares of all the errors made is evidently a quantity which can be determined in all cases. Moreover, by actual enumeration we can always have two limits between which one-half the actual deviations fall, and we can choose those limits so that their difference shall be a minimum. We may regard one-half of this difference as the probable error. We are not, however, confined to one-half as a fraction. We may take any

*American Journal of Mathematics, Volume VIII, page 343, and Elements and Constants, Chapter IV.

other fraction of the whole number, and find the closest limits between which that fraction is contained. The difference of these limits will in all cases be a fairly determinate quantity.

Table I.—*Distribution in magnitude of the proper motions $\mu \cos \delta$ of the Bradley Stars in R. A.*

	Mag. 1–2.9.				Mag. 3.0–3.9.				Mag. 4.0–4.9				Mag. 5.0–5.9.				Mag. 6.0–6.9.				Mag. 7.0–7.9.			
Quad.	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV
−17	0	6	7	0	3	7	7	1	4	7	9	0	3	8	16	2	1	14	10	5	1	4	0	0
16	...	...	...	...	...	...	...	...	1	0	0	0	0	0	3	0	2	2	1	0	...	...	...	...
15	...	...	...	...	0	1	2	0	1	1	0	0	0	3	1	0	1	2	2	1	0	1	0	1
14	0	0	1	0	1	0	0	0	...	...	...	...	...	...	...	...	0	2	1	0	...	...	...	...
13	...	...	...	...	...	...	...	...	0	0	1	1	0	1	2	0	1	1	0	0	1	1	2	0
12	...	...	...	...	0	3	0	0	0	1	1	2	0	3	4	2	2	5	0	0	0	1	1	0
11	...	...	...	...	0	1	0	0	0	2	1	0	1	3	3	2	1	3	4	1	...	...	...	...
10	0	1	3	0	0	1	1	0	2	2	0	0	1	4	3	0	1	3	2	1	...	...	...	...
9	0	0	1	0	0	0	2	0	0	3	1	0	1	2	7	1	1	2	5	0	0	0	1	0
8	0	0	1	0	1	1	1	0	1	2	4	1	1	2	3	0	1	8	5	0	0	2	3	0
7	...	...	...	...	1	2	0	1	1	1	1	1	0	6	6	0	0	4	10	1	0	2	2	1
6	...	...	...	...	1	1	2	1	0	2	2	0	4	8	8	2	7	8	9	2	1	4	1	1
5	...	...	...	...	0	3	3	0	3	2	6	0	6	4	3	6	5	11	12	2	0	4	0	0
4	1	0	0	0	2	1	5	0	3	5	1	5	3	8	12	5	6	20	15	10	2	5	1	1
3	0	0	2	1	1	5	5	1	2	2	4	5	7	9	15	2	7	14	19	11	3	7	1	5
2	0	1	1	0	0	1	2	3	1	4	7	3	11	16	12	13	16	25	23	20	1	6	1	5
1	1	0	1	1	1	2	4	2	8	12	6	6	10	19	15	13	21	34	20	26	6	9	2	4
— 0	1	2	3	0	1	0	3	2	8	6	11	12	22	22	15	18	20	40	20	23	8	10	3	5
0	1	0	0	0	0	1	0	1	0	0	2	1	5	1	4	6	3	3	1	3	0	1	0	1
+ 0	4	2	2	0	6	1	5	2	9	9	5	10	22	17	16	34	38	27	15	36	6	3	4	7
1	3	1	1	3	2	1	1	1	14	4	9	22	29	18	5	38	44	26	13	41	6	6	2	10
2	1	0	1	0	7	3	1	5	18	1	5	13	34	11	8	31	43	8	16	30	11	5	2	5
3	1	0	1	0	5	0	0	4	6	3	1	10	15	4	5	24	24	9	5	32	7	8	3	6
4	4	0	0	1	2	1	1	1	5	2	2	11	17	6	5	26	19	9	9	17	8	1	2	7
5	1	1	1	0	2	0	2	3	1	0	2	2	9	3	3	10	16	4	1	11	3	1	1	4
6	0	0	0	2	3	0	1	3	0	1	1	6	5	8	1	6	12	9	5	12	2	2	1	3
7	1	0	0	0	2	1	1	0	3	2	0	4	4	1	4	10	8	2	1	8	2	0	0	1
8	0	1	0	0	1	0	0	0	6	1	1	2	6	2	1	3	11	1	1	7	2	0	0	1
9	1	0	0	0	0	1	2	2	1	1	0	4	7	3	0	6	7	1	1	6	3	0	0	1
10	1	1	1	0	0	1	1	2	2	1	0	0	1	1	1	11	8	0	2	4	0	1	0	1
11	...	...	...	...	0	0	0	1	2	0	1	3	4	0	1	10	9	1	3	5	2	0	1	0
12	0	0	2	0	1	0	1	0	2	0	0	1	1	0	3	1	4	0	0	4	3	0	0	0
13	0	0	1	0	0	0	2	1	1	0	2	0	5	0	1	4	5	0	1	4	0	0	0	1
14	0	0	1	1	...	...	...	...	3	3	1	2	1	1	0	3	6	0	0	2	2	0	0	0
15	...	...	...	...	1	0	0	0	0	0	0	1	2	0	1	0	4	0	0	0	...	...	...	...
16	...	...	...	...	...	...	...	...	1	0	2	2	2	0	0	0	1	0	1	1	...	...	...	...
+17	4	1	0	5	6	1	1	6	7	0	3	12	15	1	7	17	12	1	6	18	4	0	0	4
	25	17	31	14	50	40	56	43	116	80	92	142	254	195	194	306	367	299	239	344	84	84	34	75

Section VI.

Statistical method of determining the precession.

I have used the numbers in Table I to ascertain what value of the precession in right ascension would result when we do not take means, but simply assume that negative and positive proper motions are equally probable, and that the most likely value of the precession is that which makes the number of positive and of negative motions equal. This determination can be made by actual count for each magnitude. But if we proceeded thus without respect to the quadrant of right ascension, we should arrive at an incorrect result on account of the effect of solar motion, combined with the unequal distribution of the stars. For example, if there are an excess of stars in

that semicircle of right ascension in which the effects of the solar motion are positive, it is evident that the result would thus be biased. We shall therefore determine, for each quadrant of right ascension, what value of the proper motion will correspond to that quadrant in such away that the number of positive and negative excesses of actual proper motion shall be equal. This value I call the mid-point. The method of finding it in the case of each quadrant and each magnitude is shown in the following tables. In any one case is given first the number of negative proper motions and then the number of positive ones. We next determine how far we must count in the direction of the excess in order to find a number of proper motions equal to one-half of that excess. In finding this point the proper motions within each space of one second have been assumed as equally distributed throughout that second.

Thus we find the mid-point for each quadrant and each magnitude as shown below:

	Magnitude 1–2.9			Magnitude 3.0–3.9		
	No. of proper motions.			No. of proper motions.		
Quadr.	Negative.	Positive.	Mid-point.	Negative.	Positive.	Mid-point.
			$''$			$''$
I	3	21	+ 3.0	12	38	+ 2.7
II	10	7	− 1.0	29	10	− 5.1
III	20	11	− 2.5	37	19	− 3 0
IV	2	12	+ 6.5	11	31	+ 3.4
Sum	35	51	+ 6.0	89	98	− 2.0

	Magnitude 4.0–4.9			Magnitude 5.0–5.9		
	No. of proper motions.			No. of proper motions.		
Quadr.	Negative.	Positive.	Mid-point.	Negative.	Positive.	Mid-point.
			$''$			$''$
I	35	81	+ 1.93	70	179	+ 2.05
II	52	28	− 1.45	118	76	− 0.90
III	55	35	− 0.87	128	62	− 2.20
IV	36	105	+ 2.11	66	234	+ 2.25
Sum	178	249	+ 1.72	382	551	+ 1.20

	Magnitude 6.0–6.9			Magnitude 7.0 +		
	No. of proper motions.			No. of proper motions.		
Quadr.	Negative.	Positive.	Mid-point.	Negative.	Positive.	Mid-point.
			$''$			$''$
I	93	271	+ 2.00	23	61	+ 2.8
II	198	98	− 1.25	56	27	− 1.4
III	158	80	− 1.94	18	16	− 0.3
IV	103	238	+ 1.70	23	51	+ 1.6
Sum	552	687	+ 0.51	120	155	+ 2.7

The mean of the four mid-points of any one group is the mid-point as it would be found were there no parallactic motion, and expresses the correction to the mean precession of all the stars in R. A. The result in the case of each magnitude for the

correction to the precession in right ascension for a century, together with the combination of the six results, is as follows:

	$''$	Wt.
Mag. 1–2	+1.5	0.1
3	−0.5	0.5
4	+0.43	1.8
5	+0.30	6.3
6	+0.13	12.0
7	+0.68	3.5
Mean:	+0.27	

I have combined the results of the several groups in yet another way by multiplying the preceding numbers of proper motions by 0.5 for magnitude 1 — 2.9; 0.7 for magnitude 3; 0.9 for magnitude 4; and unity for the remaining three magnitudes. The result is as follows:

Quad.	Negative.	Positive.	Mid-point. $''$
I	227	621	+2.11
II	444	237	−1.25
III	390	208	−1.73
IV	233	646	+2.08
	1294	1712	+1.21

Mean corr. to $\cos \delta D_t \alpha = +0''.30$

The two methods are in fair accordance, and lead to the conclusion that by the method in question the mean excess of the observed motions in the direction of R. A. above the precession N_0 is

$$+0''.285$$

This mean excess of the motion in R. A. is really the mean value of $\Delta m \cos \delta$. It must therefore be divided by the mean value of $\cos \delta$ for that part of the celestial sphere covered by the BRADLEY stars, say from the pole to 20° S. Dec. This mean we find to be by integration

$$[\cos \delta] = 0.84$$

We thus have

$$\Delta m = \frac{0''.285}{0''.84} = +0''.34$$

and

$$\delta p = \Delta m \sec \varepsilon = +0''.37$$

as the result of the statistical method.

The same numbers will enable us to determine the R. A. of the apex of the solar motion. If we put

q; the parallactic motion of a star 90° from the apex of the solar way;
A, D; the R. A. and Dec. of that apex;

In the four lines of each part of the table for each quadrant are given the numbers of the motions for each of the four magnitudes, 4, 5, 6, and 7, together with their sum.

TABLE II.—*Summation of distributed proper motions $\mu \cos \delta$ for Mags. 4 to 7.*

μ	−17″	−16″	−15″	−14″	−13″	−12″	−11″	−10″	−9″	−8″	−7″	−6″	−5″	−4″	−3″	−2″	−1″	−0″
Quad I	4	1	1	0	0	0	0	2	0	1	1	0	3	3	2	1	8	8
	3	0	0	0	0	0	1	1	1	1	0	4	6	3	7	11	10	22
	1	2	1	0	1	2	1	1	1	1	0	7	5	6	7	16	21	20
	1	0	0	0	1	0	0	0	0	0	0	1	0	2	3	1	6	8
	9	3	2	0	2	2	2	4	2	3	1	12	14	14	19	29	45	58
II	7	0	1	0	0	1	2	2	3	2	1	2	2	5	2	4	12	6
	8	0	3	0	1	3	3	4	2	2	6	8	4	8	9	16	19	22
	14	2	2	2	1	5	3	3	2	8	4	8	11	20	14	25	34	40
	4	0	1	0	1	1	0	0	0	2	2	4	4	5	7	6	9	10
	33	2	7	2	3	10	8	9	7	14	13	22	21	·38	32	51	74	78
III	9	0	0	0	1	1	1	0	1	4	1	2	6	1	4	7	6	11
	16	3	1	0	2	4	3	3	7	3	6	8	3	12	15	12	15	15
	10	1	2	1	0	0	4	2	5	5	10	9	12	15	19	23	20	20
	0	0	0	0	2	1	0	0	1	3	2	1	0	1	1	1	2	3
	35	4	3	1	5	6	8	5	14	15	19	20	21	29	39	43	43	49
IV	0	0	0	0	1	2	0	0	0	1	1	0	0	5	5	3	6	12
	2	0	0	0	0	2	2	0	1	0	0	2	6	5	2	13	13	18
	5	0	1	0	0	0	1	1	0	0	1	2	2	10	11	20	26	23
	0	0	1	0	0	0	0	0	0	0	1	1	0	1	5	5	4	5
	7	0	2	0	1	4	3	1	1	1	3	5	8	21	23	41	49	58

μ	+0″	+1″	+2″	+3″	+4″	+5″	+6″	+7″	+8″	+9″	+10″	+11″	+12″	+13″	+14″	+15″	+16″	+17″
Quad I	9	14	18	6	5	1	0	3	6	1	2	2	2	1	3	0	1	7
	22	29	34	15	17	9	5	4	6	7	1	4	1	5	1	2	2	15
	38	44	43	24	19	16	12	8	11	7	8	9	4	5	6	4	1	12
	6	6	11	7	8	3	2	2	2	3	0	2	3	0	2	0	0	4
	75	93	106	52	49	29	19	17	25	18	11	17	10	11	12	6	4	38
II	9	4	1	3	2	0	1	2	1	1	1	0	0	0	3	0	0	0
	17	18	11	4	6	3	8	1	2	3	1	0	0	0	1	0	0	1
	27	26	8	9	9	4	9	2	1	1	0	1	0	0	0	0	0	1
	3	6	5	8	1	1	2	0	0	0	1	0	0	0	0	0	0	0
	56	54	25	24	18	8	20	5	4	5	3	1	0	0	4	0	0	2
III	5	9	5	1	2	2	1	0	1	0	0	1	0	2	1	0	2	3
	16	5	8	5	5	3	1	4	1	0	1	1	3	1	0	1	0	7
	15	13	16	5	9	1	5	1	1	1	2	3	0	1	0	0	1	6
	4	2	2	3	2	1	1	0	0	0	0	1	0	0	0	0	0	0
	40	29	31	14	18	7	8	5	3	1	3	6	3	4	1	1	3	16
IV	10	22	13	10	11	2	6	4	2	4	0	3	1	0	2	1	2	12
	34	38	31	24	26	10	6	10	3	6	11	10	1	4	3	0	0	17
	36	41	30	32	17	11	12	8	7	6	4	5	4	4	2	0	1	18
	7	10	5	6	7	4	3	1	1	1	1	0	0	1	0	0	0	4
	87	111	79	72	61	27	27	23	13	17	16	18	6	9	7	1	3	51

The influence of the parallactic motion is shown very strikingly by the diversity of the extreme numbers in the four quadrants. It should evidently be allowed for in seeking the law of distribution. I therefore proceeded as follows:

Owing to the narrow limits between which the proper motions of each column are contained, we may regard the number of proper motions given in each column as

corresponding to a definite proper motion numerically greater by $0''.5$ than the number given at the top of the column. Bearing this in mind, we seek, by inspection of the sums in any one quadrant, and in each column, to find the apex of the curve whose abscissa are the proper motions and whose ordinates are the number of stars having that proper motion. I have not attempted to locate this apex with any greater precision than as being either in each column or midway between two adjacent columns. In the location I have been governed very largely by the statistical mid-points already found for the several quadrants. Thus I have been led to take—

In Quadrant I, apex between columns $1''$. and $2''$., corresponding to $\mu \cos \delta = +2''.0$;
In Quadrant II, apex between columns $-0''$. and $-1''$., corresp'g to $\mu \cos \delta = -1''.0$;
In Quadrant III, apex in column $-1''$., corresponding to $\mu \cos \delta = -1''.5$;
In Quadrant IV, apex in column $+1''$., corresponding to $\mu \cos \delta = +1''.5$.

From the data thus found I have formed the table below, in which the headings of the columns are no longer the proper motions themselves but, in the case of each quadrant, the proper motion diminished by the value corresponding to the apex of the curve just given. When, as in Quadrants III and IV, the apex is in one of the columns, I have taken the numbers as they stand in Table II, as will readily be seen by inspection. In the column of Quadrants I and II, I have taken the half sum of the corresponding numbers in each pair of adjoining columns. Moreover, I have reversed the arrangement of the numbers so that positive proper motions at the top of the column always correspond to the direction of the parallactic motion, whether this is positive or negative. Since this motion is directed very nearly from R. A. $= 270°$, the numbers of the second and third quadrants are reversed in adding them to the others.

TABLE III.—*Displaced summation for the four quadrants, II and III being turned end for end, and the apex of the curve taken for origin in each quadrant.*

$\mu \cos \delta$	$0''.0$	$1''.0$	$2''.0$	$3''.0$	$4''.0$	$5''.0$	$6''.0$	$7''.0$	$8''.0$	$9''.0$	$10''.0$	$11''.0$	$12''.0$	$13''.0$	$14''.0$	$15''.0$
Quad. I	100	79	50	39	24	18	21	22	14	14	14	11	11	9	5	40
II	76	63	41	35	30	21	18	14	10	8	8	9	6	2	4	38
III	43	43	39	29	21	20	19	15	14	5	8	6	5	1	3	39
IV	111	79	72	61	27	27	23	13	17	16	18	6	9	7	1	54
	330	264	202	164	102	86	81	64	55	43	48	32	31	19	13	171

$\mu \cos \delta$	$0''.0$	$-1''.0$	$-2''.0$	$-3''.0$	$-4''.0$	$-5''.0$	$-6''.0$	$-7''.0$	$-8''.0$	$-9''.0$	$-10''.0$
Quad. I	100	84	66	52	37	24	16	14	13	6	2
II	76	67	55	40	24	21	13	14	12	4	4
III	43	49	40	29	31	14	18	7	8	5	3
IV	111	87	58	49	41	23	21	8	5	3	1
	330	287	219	170	133	82	68	43	38	18	10

$\mu \cos \delta$	$-11''.0$	$-12''.0$	$-13''.0$	$-14''.0$	$-15''.0$	$-16''.0$	$-17''.0$	$-18''.0$
Quad. I	2	3	3	2	2	1	1	13
II	4	2	1	0	2	2	0	2
III	1	3	6	3	4	1	1	19
IV	1	1	3	4	1	0	2	7
	8	9	13	9	9	4	4	41

Some interesting conclusions are deducible from these results. One is the asymmetry of the curve representing the relation between the proper motions and the numbers. From the adoption of the apex of the curve as the zero point, it follows that, were the curves symmetrical, the numbers would diminish equally in each direction. Inspection of the table shows that, up to $4''.0$ on each side of the assumed apex, the number of negative proper motions is the greater. This shows that the adopted apex, which, from what has been said, was only roughly assumed, does not correspond to the true one, the latter being somewhere between $0''.0$ and $-1''.0$. But from $5''$. outward in each direction there is a constantly increasing excess of positive over negative proper motions. Of motions having a value of $14''.6$ and upward, 171 are positive and only 58 negative.

The cause of this asymmetry is so obvious as scarcely to need statement. It arises from the unequal distances of the stars, and the consequent inequality in the parallactic motion of the stars of any one class. The feature in question may enable us to form a rude estimate of the amount of this inequality of distance in the following way: The number of stars whose proper motion exceeds $+14''.5$ being 171, we find in the negative part of the table a proper motion $-\mu_0$, such that the number of stars having this or a greater proper motion shall be also 171. We find by count $\mu_0 = -7''.3$, as against $+14''.6$ for the stars in the opposite direction. Now, it may be assumed that the absolute values of the proper motion in the case of the 171 stars whose motion is positive is the same as in the case of the 171 stars whose motion is negative. But the difference between the two limiting values of the observed proper motions is $14''.6 - 7''.3 = 7''.3$. To this we should add the double of the parallactic motion for the apex of the curve, which is about $3''.0$. This gives $10''.3$ for twice the inferior limit of the parallactic motion of the 342 stars having the greatest absolute proper motion, and thus $5''.2$ for the limit of the parallactic motion itself. Our conclusion is that out of the 2,874 stars included in our count, about 342, or nearly one-eighth, have parallactic motions of $5''.2$ and upward.

But it must be noted that the parallactic motion thus found is not the total motion, but only the average through the whole circle of right ascension and in the direction of the right ascension. The maximum parallactic motion in right ascension for the same system of stars is found by multiplying this motion by $\pi \div 2 = 1.57$. This will carry the parallactic motion of the stars whose direction makes a right angle with the direction of the apex of solar motion up to about $8''.0$. It should be still further multiplied by sec. D to get the total absolute parallactic motion, which will then come out about $10''.0$. This will be simply the inferior limit of the motion for the group of 342 stars in question, and not the mean for all of them. At the other limit the motions correspond, approximately, to the apex of the curve. This gives us for the more distant stars of the class which we have been considering parallactic motions of $1''.5$, $2''.3$, and $3''.0$, respectively. These should be taken as the motions corresponding to the great mass of the more distant stars, and not to the limit. The latter is not capable of exact discovery, because a few scattering stars might be placed at a great distance without materially changing the result.

It is not claimed that the preceding conclusions approximate to quantitative precision. The numbers assigned are only to be considered as slight improvements on

merely qualitative statements. The fact that the apex of the curve does not approximate to a zero value of the proper motion, but to an effect of parallactic motion amounting to nearly $3''.0$, seems to show with some conclusiveness that the stars included in our count do not extend out indefinitely, but that the great mass of them are included within a comparatively limited sphere—a sphere at the boundary of which the parallactic motion would seem to be little, if any, less than $2''.0$.

In order to decide upon the best course to take in assigning weights as a function of the amount of the proper motion, we must compare the law of distribution with the theoretical distribution according to the normal law of error. A very slight examination will show that if we endeavor to represent the actual distribution by a curve following the normal law the closest representation will leave an excess of actual numbers for very small and very large values of the proper motion, while the intermediate numbers will be deficient. This tendency of distribution in residual deviations is so well known that no great interest will attach to such a comparison. What I have done is to represent the curve of distribution as closely as possible near its apex by a normal curve, and then note the deviations of the two curves as we recede from the apex in each direction. The result is shown in Table IV.

The numbers given in the first two columns of this table are taken unchanged from Table III. They therefore give, for each value of a proper motion in the direction of right ascension found in the first column, the number of stars having that value of the proper motion, the mean effect of the parallactic motion being first subtracted from the whole series in the way already described, and the positive direction being always taken in the direction of the parallactic motion.

An examination of the three central numbers of the table shows that, if we represent these numbers as abscissa of the curve of normal error, the elements of this curve will be based on a probable error of about $\pm 1''.11$, and the apex of the curve will correspond to a proper motion of about $-0''.15$. The intervals of one second, therefore, correspond to 0.90 of the probable error. Constructing a scale of abscissæ in which the probable error shall be regarded as unity, the base of the first ordinate on the negative side of the origin being placed at -0.33, and the first one on the positive side at $+0.57$, we may erect successive equidistant ordinates at the points shown in the column E of the table.

Next, from the table of the probability of errors, such IX A of CHAUVENET's *Spherical and Practical Astronomy*, Volume II, we have twice the probabilities that an error will be contained between the origin and each of the abscissæ laid down in the third column. These probabilities are found in the fourth column, P. The differences found in the column Δ are, therefore, twice the probabilities that an error will be contained between any two consecutive points on the axis.

Multiplying these numbers by the factor 694, taken arbitrarily, so as to satisfy the condition of representing the three central numbers, we have the numbers in the next column, which show the distribution of the proper motions between the several limits as it ought to be according to the normal law of error when the constants are so taken as to nearly satisfy the conditions laid down.

Lastly, the deviations from the actual numbers are given.

TABLE IV.—*Comparison of actual and normal distribution of the excesses of the proper motions of Bradley stars, in the direction of R. A., over the mean parallactic motion for magnitudes 4.0 and upward.*

$\mu \cos \delta$	Actual No.	E	P	Δ	Normal No.	Excess of actual No.
″						
—18. +	41				0	41
—17.0	4				0	4
—16.0	4				0	4
—15.0	9				0	9
—14.0	9				0	9
— 13.0	13				0	13
—12.0	9				0	9
—11.0	8				0	8
—10.0	10				0	10
— 9.0	18				0	18
— 8.0	38				0	38
— 7.0	43				0	43
		—5.73	—1.			
— 6.0	68			.0011	1	67
		—4.83	— .9989			
— 5.0	82			.0069	5	77
		—3.93	— .9920			
— 4.0	133			.0330	23	110
		—3.03	— .9590			
— 3.0	170			.1098	76	94
		—2.13	— .8492			
— 2.0	219			.2560	178	41
		—1.23	— .5932			
— 1.0	287			.4171	289	— 2
		—0.33	— .1761			
0.0	330			.4755	330	0
		+0.57	+ .2994			
+ 1.0	264			.3792	263	1
		1.47	.6786			
2.0	202			.2115	147	55
		2.37	.8901			
3.0	164			.0825	57	107
		3.27	.9726			
4.0	102			.0225	16	86
		4.17	.9951			
5.0	86			.0049	3	83
		5.07	1.			
6.0	81				0	81
7.0	64				0	64
8.0	55				0	55
9.0	43				0	43
10.0	48				0	48
11.0	32				0	32
12.0	31				0	31
13.0	19				0	19
14.0	13				0	13
15. +	171				0	171

The value of the probable error which represents that portion of the curve near the apex is thus found to be

$$\text{Probable error} = \pm 1''.11.$$

This result is so remarkable a one that I should hesitate to base any conclusions upon it were it not, as we shall soon see, completely confirmed by the distribution of the proper motions in declination. The remarkable feature is that this probable error is scarcely greater than the probable deviations of the numbers, arising from the variety of values of the parallactic motion which have been combined, and from the probable errors of the determination. That is to say, had 1,388 stars in the BRADLEY catalogue no proper motion except the parallactic motion, and were these motions determined

by observation and statistically tabulated in the manner we have done, we should expect just about such a distribution as is shown in this normal curve, the probable error $\pm 1''.11$ being the result of all the causes just mentioned. If we should now add as many more stars having all values of actual proper motion between yet wider limits and, to fix the ideas, suppose them equally distributed between these limits, say between $+15''$. and $-15''$. per century, the effect would be to rather diminish the sharpness of the curve at the apex, and therefore to increase the value of the probable error requisite to represent the central portion of the curve.

We thus see an indication that about one-half the stars in AUWERS-BRADLEY of the fourth and higher magnitudes have no appreciable proper motion, except what is due to the parallactic motion. If they have any absolute proper motion, it can be only a fraction of a second per century.

The only grounds I see on which this conclusion can be invalidated is that there may be an accidental accumulation of proper motions near the value 0. It may be remarked that in AUWERS-BRADLEY a value 0 is sometimes assigned to a proper motion simply because the data were insufficient to fix upon a definite positive or negative motion. But it must be noted that all these cases have been excluded in the count. Moreover the values which lie near the apex of the curve are not the value zero of the observed proper motions, but the values which correspond to the parallactic motion. It is quite true that in a chance distribution of about 300 quantities accidental deviations of a magnitude sufficient to change the form of the curve are quite possible. The conclusion cited must not therefore be regarded as a proven fact, but only as a conclusion indicated with a greater or less degree of probability by statistics.

SECTION VIII.

Statistical distribution of the proper motions in declination.

I have not studied the proper motions in declination by the statistical method so fully as those in right ascension, because it is not so easy to deduce results from them. I have, however, tabulated the distribution of the proper motions in declination as given by AUWERS without the application of my systematic corrections. As the uncorrected values will answer all the purposes now in view, the results and comparison are taken from a paper in the *Astronomical Journal* and tabulated below.

As to magnitude, the stars were divided into four groups as follows:

Group I, Mag. 3.4 and brighter;
" II, " 3.5 to 4.9;
" III, " 5.0 to 5.9;
" IV, " 6.0 and upward.

It will be seen that there is an excess of proper motions between $0''$. and $-1''$. per century defining the apex of the curve, and as well marked as in the case of the right ascensions. The negative position of the apex is evidently due to the parallactic motion in declination.

TABLE V.—*Distribution of* AUWERS'S *centennial proper motions in declination, according to magnitude, with probable distributions according to the normal law of error.*

Limits.	Actual distribution.					Probable distribution.					
						$p.\ e. = \pm 2''.53$		$p.\ e. = \pm 1''.50$		$p.\ e. = 1''.11$	
	Group I, 1^m to $3^m.4$.	Group II, $3^m.5$ to $4^m.9$.	Group III, 5^m to $5^m.9$.	Group IV, $6^m.0+$	Total.	No.	Excess.	No.	Excess.	No.	Excess.
″ ″											
−∞ and −15	24	52	50	49	175	0	175	0	175	0	175
−15 " −14	3	2	3	7	15	0	15	0	15	0	15
−14 " −13	3	1	8	9	21	1	20	0	21	0	21
−13 " −12	1	2	4	12	19	2	17	0	19	0	19
−12 " −11	2	4	6	9	21	5	16	0	21	0	21
−11 " −10	4	8	11	16	39	11	28	0	39	0	39
−10 " − 9	2	6	7	25	40	21	19	0	40	0	40
− 9 " − 8	4	6	12	21	43	36	7	0	43	0	43
8 " − 7	6	9	20	16	51	62	−11	2	49	0	51
− 7 " − 6	6	13	32	34	85	98	−13	11	74	0	85
− 6 " − 5	4	14	26	54	98	143	−45	32	66	6	92
− 5 " − 4	3	21	35	72	131	197	−66	80	51	26	105
− 4 " − 3	19	32	60	129	240	251	−11	158	82	88	152
− 3 " − 2	15	35	79	147	276	299	−23	257	19	213	63
− 2 " − 1	16	50	107	184	357	332	25	344	13	356	1
− 1 " 0	22	71	121	211	425	342	83	377	48	417	8
0 " + 1	21	30	98	192	341	330	11	341	0	342	− 1
+ 1 " + 2	19	30	72	158	279	297	−18	249	30	196	83
+ 2 " + 3	12	32	51	92	187	248	−61	153	34	79	108
+ 3 " + 4	8	22	29	48	107	193	−86	75	32	22	85
+ 4 " + 5	3	16	19	32	70	140	− 70	31	39	5	65
+ 5 " + 6	5	4	17	20	46	94	−48	11	35	0	46
+ 6 " + 7	2	1	9	14	26	62	−36	2	24	0	26
+ 7 " + 8	2	6	3	11	22	35	− 13	0	22	0	22
+ 8 " + 9	1	6	4	10	21	19	2	0	21	0	21
+ 9 " +10	2	0	8	3	13	9	4	0	13	0	13
+10 " +11	0	2	4	5	11	4	7	0	11	0	11
+11 " +12	0	2	4	1	7	2	5	0	7	0	7
+12 " ∞	12	18	20	15	65	1	64	0	65	0	65
Sums.	221	495	919	1,596	3,231	3,234	....	2,123	1,108	1,750	1,481

SECTION IX.

Formation and solution of equations from individual stars.

The very simple formulæ for the equations of condition between the four unknown quantities have already been quoted. Preliminary to the formation of the equations, all the stars of AUWERS-BRADLEY were arranged according to their magnitudes as given in AUWERS's work. All stars up to magnitude 2.9 were included in one group, because they are too few in number to admit of any discussion of smaller groups. Stars of magnitude 3.0 to 3.9 were tabulated as of magnitude 3, and so with the following magnitudes up to 7, which included all stars of the 7th and higher magnitudes. Separate normal equations were then formed for each of the magnitudes.

The study of the statistical distribution of the proper motions just made shows that we here have a case where the assignment of weights as a function of the deviation on the principles already developed by me might be applicable. I have, however, felt that not enough would be gained by the application to justify the investigation it would involve. I have therefore adopted the usual method of simply excluding all proper

excluded in investigations on the subject. In fact, all the stars remaining in the equation will be among those hitherto classified as having small proper motions.

The limits have been set as a function of the right ascension in order that the effect of the parallactic motion might be included in determining them in each separate case. Properly speaking, the limits should not be those of the absolute terms in the equation of condition, or the absolute proper motion, but should be determined by the residuals left after the solution of the equations of condition. The limits thus derived for the right ascension are given in the following Table VI:

TABLE VI.—*Limits outside of which proper motions in R. A. are excluded.*

R. A.	Mag. 1–2	3	4	5–7
h	″ ″	″ ″	″ ″	″ ″
0	−12 to +24	− 9+18	− 7+15	− 6+12
1	−12 +24	− 9 18	− 7+15	− 6+12
2	−13 +23	−10 17	− 8+14	− 7+11
3	−14 +23	−11 16	− 9+13	− 7+11
4	−15 +22	−12 15	− 9+12	− 8+11
5	−16 +20	−12 15	−10+11	− 8+10
6	−17 +19	−13 14	−11+11	− 9+ 9
7	−20 +16	−14 13	−12+10	−10+ 8
8	−22 +15	−15 12	−12+ 9	−10+ 8
9	−22 +14	−17 11	−13+ 8	−11+ 7
10	−23 +13	−17 10	−14+ 8	−12+ 7
11	−24 +12	−18 9	−14+ 8	−12+ 6
12	−24 +12	−18 9	−14+ 7	−12+ 6
13	−24 +12	−18 9	−14+ 8	−12+ 6
14	−23 +13	−17 10	−13+ 8	−11+ 7
15	−22 +14	−16 11	−13+ 9	−11+ 7
16	−21 +15	−16 11	−13+ 9	−11+ 8
17	−20 +16	−15 12	−12+ 9	−10+ 8
18	−17 +19	−14 13	−12+10	− 9+ 9
19	−16 +21	−13 14	−11+10	− 8+10
20	−15 +22	−12 15	−10+11	− 8+10
21	−14 +23	−11 16	− 9+12	− 7+11
22	−13 +23	−10 17	− 9+13	− 7+12
23	−12 +24	− 9 18	− 8+14	− 6+12

In the case of the declinations the limits were of course the same, except that the allowance for parallactic motion was different and was generally smaller than in the case of the right ascensions.

The general rule followed was to exclude a star if its proper motion in either coordinate was outside the limits of the table. It is, indeed, clear enough that since the proper motion has no relation to coordinates the exclusion should be determined by the absolute amount of the proper motion, irrespective of the question whether it chance to coincide with the direction of one of the coordinates. For this reason the rules of exclusion were not entirely absolute, but were in general about these: Stars markedly outside the limit in the case of either coordinate were always excluded; if, however, nearly on the limit, they would be included when the motion in the other coordinate was small. Moreover, if there were several stars in any region of the heavens near the same positive or negative limit of exclusion, some would be included and others excluded. When the motions in both coordinates approximated to the limits, though included within them, the star was excluded.

It appears by count that the number of excluded stars and the total number in the several classes are as follows:

	Number of stars. Total.	Excluded.
Mag. 1 and 2	87	23
" 3	189	54
" 4	430	103
" 5	949	218
" 6	1, 249	215
" 7	277	41
	3, 181	654

It will be seen that a little more than one-fifth the entire number of stars are excluded.

SECTION X.

Formation of the normal equations.

In forming the normal equations I employed in most cases the method of equivalent factors proposed by G. P. BOND in the Memoirs of the American Academy of Arts and Sciences about the middle of the present century, but rarely applied. It is in some sort an intermediate method between the rigorous treatment by least squares and the system sometimes used of forming normal equations from conditional equations by using no factors except $+1$, 0, or -1 to form each normal equation. Though we may call the method intermediate, it is not to be regarded as midway between the two systems, as its deviation from rigor is practically inappreciable.

The method amounts, in fact, to allowing a very slight change in the weights with which the several conditional equations enter into the normal equations. In forming the latter each conditional equation is multiplied by the product of the coefficient of the corresponding unknown quantity into the weight of the equation. Instead of using the rigorous value of this factor we use some convenient quantity near it, say the nearest integer or the nearest tenth. The normal equation is the sum of the products of the conditional equations by these slightly modified factors. In this method the factors $[ab]$ and $[ba]$ are not rigorously equal, and must therefore be formed separately.

In the present case I have taken as the factors ten times the coefficient of each unknown quantity in each normal equation. Thus we have the following normal equations and solutions from the stars of each class:

Normal equations and solutions for the Bradley stars.

Mag. 1–2.9

R. A.

$$\begin{array}{rrrrl}
402\,\delta p & +\ 31\,X & +\ 19\,Y & = & +\ 753'' \\
+30 & +350 & -\ 77 & = & +\ 156 \\
+18 & -\ 77 & +301.4 & = & -1604 \\
 & & \delta p & = & +2.2 \\
 & & X & = & -1.0 \\
 & & Y & = & -5.7
\end{array}$$

Dec.

$$\begin{array}{rrrrrl}
47\,\delta p & +\ 8\,Y & +\ 49\,X & +\ 11\,Z & = & +\ 10'' \\
+\ 8 & +80 & +\ 21 & 0 & = & -\ 542 \\
+49 & +21 & +107 & -\ 18 & = & -\ 118 \\
+11 & 0 & -\ 18 & +453 & = & +1601 \\
 & & & \delta p & = & -0.8 \\
 & & & X & = & +1.3 \\
 & & & Y & = & -7.0 \\
 & & & Z & = & +3.6
\end{array}$$

Normal equations and solutions for the Bradley stars—**Continued.**

Mag. 3.0–3.9

R. A.

$$\begin{array}{rrrrl} & & & & '' \\ 904\,\delta p & +100\,X & -15\,Y & = & +278 \\ +100 & +850 & -46 & = & -98 \\ -15 & -46 & +501 & = & -2609 \\ & & \delta p & = & +0.27 \\ & & X & = & -0.43 \\ & & Y & = & -5.21 \end{array}$$

Dec.

$$\begin{array}{rrrrrl} & & & & & '' \\ 78\,\delta p & +3\,Y & +60\,X & -6\,Z & = & -93 \\ +3 & +173 & +13 & -8 & = & -743 \\ +60 & +13 & +138 & +31 & = & -191 \\ -6 & -8 & +31 & +1039 & = & +2595 \\ & & & \delta p & = & +0.50 \\ & & & X & = & -1.78 \\ & & & Y & = & -4.04 \\ & & & Z & = & +2.52 \end{array}$$

Mag. 4.0–4.9

$$\begin{array}{rrrrl} & & & & '' \\ 2165\,\delta p & +12\,X & -404\,Y & = & +1727 \\ +18 & +1822 & +39 & = & -378 \\ -405 & +45 & +1485 & = & -4658 \\ & & \delta p & = & +0.23 \\ & & X & = & -0.14 \\ & & Y & = & -3.07 \end{array}$$

$$\begin{array}{rrrrrl} & & & & & '' \\ 240\,\delta p & +10\,Y & +129\,X & -176\,Z & = & +39 \\ +12 & +399 & -2 & +42 & = & -886 \\ +127 & -1 & +354 & -77 & = & -94 \\ -171 & +44 & -80 & +2535 & = & +4291 \\ & & & \delta p & = & +1.72 \\ & & & X & = & -0.49 \\ & & & Y & = & -2.47 \\ & & & Z & = & +1.84 \end{array}$$

Mag. 5.0–5.9

$$\begin{array}{rrrrl} & & & & '' \\ 4793\,\delta p & +472\,X & -653\,Y & = & +2880 \\ +468 & +3926 & +268 & = & -388 \\ -660 & +270 & +3440 & = & -9525 \\ & & \delta p & = & +0.22 \\ & & X & = & +0.06 \\ & & Y & = & -2.73 \end{array}$$

$$\begin{array}{rrrrrl} & & & & & '' \\ 555\,\delta p & -45\,Y & +472\,X & -318\,Z & = & +334 \\ -54 & +912 & -129 & -152 & = & -2448 \\ +458 & -125 & +1031 & -287 & = & -81 \\ -308 & -142 & -282 & +5322 & = & +8804 \\ & & & \delta p & = & +2.00 \\ & & & X & = & -0.83 \\ & & & Y & = & -2.68 \\ & & & Z & = & +1.58 \end{array}$$

Mag. 6.0–6.9

$$\begin{array}{rrrrl} & & & & '' \\ 7155\,\delta p & +897\,X & -403\,Y & = & +2611 \\ +895 & +5258 & +283 & = & +1555 \\ -410 & +260 & +5144 & = & -13552 \\ & & \delta p & = & +0.16 \\ & & X & = & +0.42 \\ & & Y & = & -2.77 \end{array}$$

$$\begin{array}{rrrrrl} & & & & & '' \\ 835\,\delta p & -30\,Y & +667\,X & -211\,Z & = & +1100 \\ -36 & +1236 & -92 & -678 & = & -2798 \\ +646 & -93 & +1397 & -316 & = & +752 \\ -199 & -668 & -297 & +7763 & = & +12454 \\ & & & \delta p & = & +1.56 \\ & & & X & = & +0.14 \\ & & & Y & = & -1.01 \\ & & & Z & = & +1.56 \end{array}$$

Mag. 7.0–

$$\begin{array}{rrrrl} & & & & '' \\ 1637\,\delta p & +496\,X & -114\,Y & = & +961 \\ +492 & +1310 & +116 & = & +286 \\ -119 & +117 & +1065 & = & -2451 \\ & & \delta p & = & +0.34 \\ & & X & = & +0.29 \\ & & Y & = & -2.30 \end{array}$$

$$\begin{array}{rrrrrl} & & & & & '' \\ 172\,\delta p & -11\,Y & +160\,X & -58\,Z & = & +200 \\ -11 & +356 & -32 & -213 & = & -956 \\ +154 & -24 & +314 & -53 & = & +300 \\ -59 & -214 & -50 & +1709 & = & +3047 \\ & & & \delta p & = & +1.06 \\ & & & X & = & +0.59 \\ & & & Y & = & -1.62 \\ & & & Z & = & +1.63 \end{array}$$

SECTION XI.

Combination of the partial normal equations into a single system.

I do not regard the results of the preceding solutions as those which are to be finally combined in a definitive result. Separate solutions for the different magnitudes have been made in order to ascertain whether there is any systematic difference between the precessional motion as derived from stars of different magnitudes, or as derived from right ascensions and declinations, respectively.

There is evidently a marked excess in the value of δp derived from the declinations over that derived from the right ascensions. This discrepancy will be discussed

subsequently. There is no evidence of a progressive change with the magnitude. It is true that the results from the few brighter stars are markedly discordant; but this is to be expected, not only from the small number of the stars, but from their considerable miscellaneous proper motions. When the results from the right ascensions and from the declinations are combined, it will be seen that the discordance is materially diminished.

We might suspect from the first four values of δp, derived from the declinations, that there is a progressive increase with the magnitude. But we find that this does not continue through the two next higher magnitudes. On the whole we can not say that there is any evidence of well-marked progressive change with magnitude. We shall therefore in the subsequent solutions combine all the magnitudes. The factors with which this is to be done are next to be considered.

What we should really do is to multiply the conditional equations by such factors (square roots of the weights) as will reduce the probable errors of the several equations to the same value. But these probable errors are nothing more than the miscellaneous deviations of the proper motions. It follows that if we form a series of numbers inversely proportional to these miscellaneous residual proper motions for the several magnitudes of the stars, such numbers will form the appropriate factors of our equations when the latter are written in the form

$$a\,\delta p + \frac{\sin\alpha}{\rho}X - \frac{\cos\alpha}{\rho}Y = \mu\cos\delta$$
$$b\,\delta p + \frac{\cos\alpha\sin\delta}{\rho}X + \frac{\sin\alpha\sin\delta}{\rho}Y - \frac{\cos\delta}{\rho}Z = \mu'$$

We have still to consider the divisor ρ which enters into the equations and expresses the distances of the several classes of stars. There being no method of determining ρ, we are obliged to assume that for each class of stars it is inversely proportional to the mean deviations of the proper motions, and therefore directly proportional to the factors just defined. It follows that if we write the equations in the form

$$a\,\rho\,\delta p + \sin\alpha\,X - \cos\alpha\,Y = \mu\,\rho\cos\delta = \rho\,n$$
$$b\,\rho\,\delta p + \cos\alpha\sin\delta\,X + \sin\alpha\sin\delta\,Y - \cos\delta\,Z = \mu'\,\rho = \rho\,n'$$

the equations will all be of equal weight. We must therefore determine the factors ρ so that they shall be approximately the inverse of the proper motions. We might equally well take the parallactic motion as determining ρ for each class of stars. What I have actually done is to find the deviations in various ways, and to consider also the conclusions of others on the subject. FEDORENKO (Astr. Nach. Vol. XLV, pp. 81–84) long since found that the proper motions of stars from magnitude 1 to 8 were nearly the inverse of the magnitudes. This approximate law was confirmed by AUWERS in his discussion of the proper motions of the stars contained in his Berlin zone. We can scarcely regard this law as a real law of nature. Its most obvious defect is that it would make the proper motions of stars of the magnitude 0, infinite. We must therefore expect that, in the case of the higher magnitudes, as the first and second, the law will give the proper motions too large. Again, in the case of the very

faint stars the proper motions included are not the mean proper motions of all stars, but only of a selected number. When all stars are included we must concede that the product will be less than that given by the rule as fainter stars are taken. But the rule is probably very nearly true for the magnitudes with which we are concerned, especially those from the fourth to the seventh. Comparing the rule with the general probabilities of the case, with the average deviations shown by the statistical distribution of the proper motions and the several values of the parallactic motion, I have reached the conclusion that the following values of ρ form as good an approximation to the required factors as we can readily obtain:

Mags.	1–2.9;	$\rho = 0.4$
"	3.0–3.9;	$= 0.6$
"	4.0–4.9;	$= 0.8$
"	5.0–5.9;	$= 1.0$
"	6.0–6.9;	$= 1.2$
"	7.0–	$= 1.4$

From the equations last written it will be seen that in forming the normal equations we should in each partial normal equation already given multiply

$$[a\,a],\ [a\,n],\ [b\,b],\ [b\,n]\quad .\ .\ .\ .\quad \text{by } \rho^2$$

and the other products by ρ, or by unity, according to the degree to which ρ enters into the product of each pair of coefficients in the equations.

When the coefficients of the sets of normal equations found on pages 31–32 are multiplied by these factors and combined into a single set, we have the following results. The right ascensions and declinations are first taken separately; then they are combined by addition:

Combined equations from all Bradley stars.

R. A.

$$\begin{array}{rrrr}
20079\delta p + & 2324X - & 1621Y = & +\ 9849'' \\
2317 & +13516 & +\ 583 & =+\ 1579 \\
-\ 1645 & +\ 570 & +11937 & =-35151
\end{array}$$

Declination.

$$\begin{array}{rrrrr}
2284\delta p - & 82Y + & 1654X - & 792Z = & +\ 2302'' \\
-\ 100 & +3157 & -\ 221 & -\ 1009 & =-\ 8516 \\
+1605 & -\ 209 & +3341 & -\ 720 & =+\ 1004 \\
-\ 764 & -\ 988 & -\ 696 & +18821 & =+33644
\end{array}$$

R. A. and Dec. combined.

$$\begin{array}{rrrrr}
22363\delta p + & 3978X - & 1703Y - & 792Z = & +12151'' \\
3922 & +16857 & +\ 374 & -\ 720 & =+\ 2583 \\
-1745 & +\ 349 & +15094 & -\ 1009 & =-43667 \\
-\ 764 & -\ 696 & -\ 988 & +18821 & =+33644
\end{array}$$

The following are the solutions which we obtain by treating the three sets independently:

	R. A.	Dec.	Combined.
	″	″	″
$\delta p=$	$+0.231$	$+1.748$	$+0.357$
$X=$	$+0.203$	-0.297	$+0.203$
$Y=$	-2.920	-2.108	-2.745
$Z=$	. . . .	$+1.737$	$+1.666$

As there can be but one value of the several quantities X, Y, and Z for any given system of stars, the final values of these quantities are those which we should use in obtaining the value of δp from each separate partial normal equation in δp. From the factors we have employed it will be seen that the preceding values correspond to magnitude 5.5. We reduce them to the other magnitudes by dividing by the same factors which we used in the multiplication, and thus obtain the following results:

Values of X, Y, *and* Z *for the several groups.*

Mag.	1 – 2.9	3.5	4.5	5.5	6.5	7.+
	″	″	″	″	″	″
X	$+0.51$	$+0.34$	$+0.25$	$+0.203$	$+0.17$	$+0.14$
Y	-6.86	-4.58	-3.43	-2.745	-2.29	-1.96
Z	$+4.16$	$+2\ 78$	$+2.08$	$+1.666$	$+1.39$	$+1.19$

We now substitute these values of X, Y, Z in each partial normal equation in δp, whether in right ascension, declination, or the combination of both, in the combined equations from all the right ascensions, and in the combined equations from all the declinations. We thus have the following separate results for δp from the stars of each separate magnitude in right ascension and in declination, and from the combination of both motions, together with the final results from all the right ascensions, all the declinations, and the combination of both coordinates.

Separate and combined results when the definitive values of X, Y, and Z are substituted in the normal equations in δp.

	R. A.	Dec.	Both.
	″	″	″
Mag. 1 – 2	$\delta p=+2.16$	$\delta p=-0.10$	$\delta p=+1.92$
3	$+0.19$	-1.05	$+0.09$
4	$+0.16$	$+1.70$	$+0.19$
5	$+0.21$	$+1.16$	$+0.31$
6	$+0.22$	$+1.35$	$+0.35$
7	$+0.41$	$+1.30$	$+0.53$
All	$+0.245$	$+1.340$	$+0.357$

The results are as accordant as we could expect when derived from the stars of the separate magnitudes. But there is clearly a systematic difference between the results derived from the right ascensions and those from the declinations. In the case of the latter the result is the more noteworthy from its discordance with that of

L. STRUVE, who, from substantially the same data, obtained a result which, when made comparable with the above, would imply a negative correction to the results derived from all the declinations. As a prelude to considering the cause of the discrepancy, and to deriving the most probable combination, we have to consider the possible effect of a community of proper motions among many stars in different regions of the heavens. In connection with this the consistency between the results derived from the stars in different directions will have to be considered.

SECTION XII.

Classification of the stars by regions and zones.

The results obtained in the preceding section have been derived by giving each individual star equal weight. But it is well-known that a community of proper motions frequently exists among small groups of stars—the Pleiades, for example. It is evident that in such cases the entire group should receive only the weight of a single star, or perhaps two. Were we to seek out the special cases of this kind, so far as they could be determined from the equations, they would be so few as not to materially affect our result. What I have therefore done is to divide the heavens into regions, each forming a trapezium of one or two hours of right ascension and extending through 15° or 30° of declination, and find the mean proper motion of all the stars in each trapezium. A process somewhat similar to this was followed by L. STRUVE in combining his equations from the separate stars. From the mean of all the stars in each trapezium he formed a single equation of condition, assuming the mean proper motion to correspond to the mean position. He then gave to each equation a result proportional (making abstraction of differences of magnitude) to the number of stars which the trapezium contained. Thus, his results are the same as if he had found and solved the conditional equations for each star separately, as I have done in the preceding section. I consider the right ascensions and declinations separately.

SECTION XIII.

Discussion of the right ascensions.

When equations are combined, on the supposition of a possible community of proper motions within each region, the question of combining stars of different magnitudes offers some difficulty. Such a common motion presupposes that the stars are nearly at the same distance, and therefore have approximately the same proper motion. It would seem, therefore, that in such a case all stars should be given equal weight, irrespective of their magnitude. But it is certain that there are many scattered stars to which the hypothesis would not apply. What I have, therefore, done in the case of the R. A.'s is to confine the discussion to stars of magnitudes 4.0 and upward. Equal weight was then given to each star, so that the mean proper motion within each zone and region is that of the stars in question within the region. The zones are 15° wide, except in the case of that south of — 15°. Here I have included all stars as if belonging to a single zone. The mean proper motions thus derived from the second members of the equations of condition are written in the table below. For example, in

TABLE VII.—*Equations for precession from proper motion in R. A. by regions, etc.*—**Continued.**

R. A.	Zone +15° to +30°						Zone +30° to +45°					
	$\delta = +22\frac{1}{2}°$						$\delta = 37\frac{1}{2}°$					
°	p	x	y		$''$	Wt.	p	x	y		$''$	Wt.
7½	.86	+.13	−.99	=	$+3.0_{23}$	7	.76	+.13	−.99	=	$+0.5_{13}$	6
22½	.90	+.38	−.92	=	2.8_{26}	7	.82	+.38	−.92	=	1.4_{12}	6
37½	.94	+.61	−.79	=	2.3_{28}	8	.88	+.61	−.79	=	2.9_{12}	6
52½	.97	+.79	−.61	=	1.8_{35}	8	.92	+.79	−.61	=	1.0_{9}	5
67½	.99	+.92	−.38	=	4.1_{35}	8	.95	+.92	−.38	=	1.4_{15}	6
82½	1.00	+.99	−.13	=	1.9_{40}	8	.97	+.99	−.13	=	0.8_{16}	6
97½	1.00	+.99	+.13	=	$+0.5_{36}$	8	.97	+.99	+.13	=	0.5_{12}	6
112½	.99	+.92	+.38	=	-1.1_{34}	8	.95	+.92	+.38	=	$+0.9_{10}$	5
127½	.97	+.79	+.61	=	-2.4_{33}	8	.92	+.79	+.61	=	-0.9_{14}	6
142½	.94	+.61	+.79	=	-2.4_{14}	6	.88	+.61	+.79	=	-0.8_{14}	6
157½	.90	+.38	+.92	=	-3.2_{16}	6	.82	+.38	+.92	=	-3.3_{22}	7
172½	.86	+.13	+.99	=	-1.9_{12}	6	.76	+.13	+.99	=	-5.0_{2}	2
187½	.82	−.13	+.99	=	-2.9_{29}	8	.70	−.13	+.99	=	-2.7_{10}	5
202½	.79	−.38	+.92	=	-2.5_{10}	5	.63	−.38	+.92	=	-7.8_{4}	3
217½	.76	−.61	+.79	=	-1.1_{8}	5	.58	−.61	+.79	=	-10.8_{1}	1
232½	.72	−.79	+.61	=	-3.6_{18}	7	.54	−.79	+.61	=	-1.8_{12}	6
247½	.71	−.92	+.38	=	-1.4_{15}	6	.50	−.92	+.38	=	-1.9_{15}	6
262½	.70	−.99	+.13	=	$+0.2_{15}$	6	.48	−.99	+.13	=	-1.3_{8}	5
277½	.70	−.99	−.13	=	0.8_{11}	6	.48	−.99	−.13	=	$+0.4_{19}$	7
292½	.71	−.92	−.38	=	1.3_{39}	8	.50	−.92	−.38	=	$+0.9_{14}$	6
307½	.72	−.79	−.61	=	0.6_{21}	7	.54	−.79	−.61	=	2.0_{23}	7
322½	.76	−.61	−.79	=	3.4_{17}	6	.58	−.61	−.79	=	1.0_{12}	6
337½	.79	−.38	−.92	=	0.7_{7}	4	.63	−.38	−.92	=	3.4_{13}	6
352½	.82	−.13	−.99	=	$+2.5_{16}$	6	.70	−.13	−.99	=	$+3.5_{11}$	6

R. A.	Zone +45° to +60°						Zone +60° to +90°					
	$\delta = 52\frac{1}{2}°$						$\delta = 70°$					
°	p	x	y		$''$	Wt.	p	x	y		$''$	Wt.
7½	.60	+.13	−.99	=	$+1.6_{13}$	6	.36	+.13	−.99	=	$+2.3_{21}$	7
22½	.68	+.38	−.92	=	2.2_{15}	6	.46	+.38	−.92	=	2.6_{25}	7
37½	.74	+.61	−.79	=	1.6_{10}	6	.54	+.61	−.79	=	0.1_{18}	7
52½	.80	+.79	−.61	=	3.0_{11}	6	.61	+.79	−.61	=	2.3_{4}	3
67½	.85	+.92	−.38	=	1.0_{11}	6	.66	+.92	−.38	=	$+0.1_{2}$	2
82½	.87	+.99	−.13	=	$+2.0_{19}$	7	.68	+.99	−.13	=	-0.4_{9}	5
97½	.87	+.99	+.13	=	-0.9_{10}	5	.68	+.99	+.13	=	-0.6_{10}	5
112½	.85	+.92	+.38	=	-0.1_{12}	6	.66	+.92	+.38	=	$+0.1_{11}$	6
127½	.80	+.79	+.61	=	$+3.5_{5}$	4	.61	+.79	+.61	=	-0.5_{9}	5
142½	.74	+.61	+.79	=	$+0.3_{6}$	4	.54	+.61	+.79	=	-4.2_{8}	5
157½	.68	+.38	+.92	=	-1.6_{6}	4	.46	+.38	+.92	=	-2.2_{6}	4
172½	.60	+.13	+.99	=	-1.3_{7}	4	.36	+.13	+.99	=	-0.2_{2}	2
187½	.52	−.13	+.99	=	-0.1_{9}	5	.27	−.13	+.99	=	-1.9_{11}	6
202½	.44	−.38	+.92	=	-2.1_{8}	5	.17	−.38	+.92	=	0.0_{1}	1
217½	.36	−.61	+.79	=	-2.9_{5}	4	.09	−.61	+.79	=	-2.8_{3}	2
232½	.31	−.79	+.61	=	$+6.8_{1}$	1	+.02	−.79	+.61	=	$+2.4_{2}$	2
247½	.26	−.92	+.38	=	-2.4_{6}	4	−.03	−.92	+.38	=	-1.1_{5}	4
262½	.24	−.99	+.13	=	-1.0_{5}	4	−.06	−.99	+.13	=	-2.1_{3}	2
277½	.24	−.99	−.13	=	$+0.6_{8}$	5	−.06	−.99	−.13	=	-0.2_{11}	6
292½	.26	−.92	−.38	=	$+0.1_{8}$	5	−.03	−.92	−.38	=	$+0.4_{8}$	5
307½	.31	−.79	−.61	=	1.6_{15}	6	+.02	−.79	−.61	=	1.1_{19}	7
322½	.36	−.61	−.79	=	2.7_{11}	6	.09	−.61	−.79	=	1.2_{18}	7
337½	.44	−.38	−.92	=	1.6_{13}	6	.17	−.38	−.92	=	1.9_{15}	6
352½	.52	−.13	−.99	=	$+2.4_{17}$	7	+.27	−.13	−.99	=	$+1.6_{22}$	7

By the third process we should find a result substantially the same as that already derived by the solutions from the individual stars. The deviations which would arise from the center of the trapezium not always coinciding accurately with the center of

gravity of the stars and from the different magnitudes having equal weight would be likely to produce only a small change in the result.

In order to bring out the differences sought for, I have made a solution for each zone separately.

In the solution based on equal weights for each trapezium I have omitted the extreme southern zone, owing to the irregular distribution of the stars, and also the northern one, owing to its small weight. To eliminate the parallactic motion I have simply added the equations in each of the other zones with the results:

Zone	-15° to 0°;	$21.8\,\delta p =$	$-\ 0.2''$
	0° to $+15^\circ$;	21.8	$+\ 7.1$
	$+15^\circ$ to $+30^\circ$;	20.3	$+\ 3.4$
	$+30^\circ$ to $+45^\circ$;	17.4	$+\ 3.3$
	$+45^\circ$ to $+60^\circ$;	13.4	$+18.1$

From these we derive the normal equation

$$1850\,\delta p = +520''.$$

Whence,

$$\delta p = +0''.28.$$

The second or adjusted system of weights gives rise to the following normal equations:

$$\begin{aligned} 598.7\,\delta p + 58.1X - 33.7Y &= +225.0'' \\ 476.6 \quad + 17.9 \quad &= +76.1 \\ 456.8 \quad &= -1231.2 \end{aligned}$$

These give

$$\begin{aligned} \delta p &= +0.20'' \\ X &= +0.24 \\ Y &= -2.69 \end{aligned}$$

This I regard as the preferable result of the method, and have therefore made a comparison of the partial results from the seven separate zones of Dec. This is done by taking the partial normal equation in p given by each zone, and substituting the concluded values of X and Y so as to derive a value of δp. The results are as follows:

Zones.	Equations.					
		$''$			$''$	$''$
-37° to -25°	$93.4\,\delta p =$	$36.4 + 30.8X$	$+\ 1.4Y$	$=+$	40.0;	$\delta p = +\ 0.43$
-15° to 0°	123.2	$= 26.4 +\ 2.8$	$+\ 5.9$	$=+$	11.2	$+\ 0.09$
0° to $+15^\circ$	121.4	$= 51.9 -\ 8.3$	$+\ 4.9$	$=+$	36.7	$+\ 0.30$
$+15^\circ$ to $+30^\circ$	121.0	$= 38.0 - 21.5$	$+\ 1.9$	$=+$	27.7	$+\ 0.23$
$+30^\circ$ to $+45^\circ$	74.1	$= 10.3 - 17.7$	$+\ 8.3$	$=-$	16.2	$-\ 0.22$
$+45^\circ$ to $+60^\circ$	46.3	$= 60.2 - 23.2$	$+\ 7.9$	$=+$	33.3	$+\ 0.72$
$+60^\circ$ to $+80^\circ$	19.3	$=\ 1.8 - 21.0$	$+\ 3.4$	$=-$	12.3	$-\ 0.64$
Total,	$598.7\,\delta p =$	$225.0 - 58.1$	$+33.7$	$=+$	120.4;	$\delta p = +0.201$

By the third system, in which the weights are proportional to the number of stars, we have the following normal equations and solutions:

$$\begin{array}{rrrl} 1568p + & 187X - & 168Y & = + 796'' \\ & 1209 - & 17 & = + 231 \\ & & 1086 & = - 2{,}975 \end{array}$$

$$\begin{aligned} X &= +0''.12 \\ Y &= -2.71 \\ p &= +0.204 \end{aligned}$$

This is less by $0''.04$ than the result from each star separately, a difference which arises partly from the fact that in this solution stars of all magnitudes have equal weights. If we combine the four results on page 35, for the magnitudes 4 to 7, namely, $''.16$, $''.21$, $''.22$, $''.41$, giving each a weight proportional to the number of stars, we have $0''.225$. We thus have the following three sets of values for p by the three separate systems of treatment:

All equal weights, ...	$p = +0''.28$
Adjusted weights, ...	$+0.20$
Separate stars, ...	$+0.24$

These differences are so slight as to show that the result will not be materially altered by allowing for a supposed community of proper motions in different regions. The slight deviations from the former result seem rather to increase the deviation from the result of the declinations.

A singular feature is, however, shown by the results when we arrange them by right ascensions. It is not possible to give a definite result for the precession from each separate hour of right ascension, owing to the effect of the parallactic motion. But from the mean result of any two opposite hours of right ascension the parallactic motion may be eliminated, always supposing that the stars of equal apparent magnitudes are at equal distances. I have proceeded in the following way to make a comparison of mean proper motions derived from regions of equal and opposite values of the parallactic motion. Four regions of right ascension are included as follows: Taking the right ascension A of the solar apex as 18 h. 30 m. the adopted regions are lunes designated as (a) (b) (c) and (d).

(a) Right ascensions within $2\frac{1}{2}$ hours $\pm$ from $A + 90°$;
(b) " " " " " $A + 180°$;
(c) " " " " " $A - 90°$;
(d) " " " " " A.

It will be seen that each of these regions includes five hours of right ascension, leaving a space of one hour between adjacent regions. Within the four regions the mean proper motions are taken, assigning to those within each trapezium the weights

employed in the preceding equations of condition. The results for the mean proper motions in the direction of R. A. are:

Zone.		(a.)	(b.)	(c.)	(d.)	½(a+c)	½(b+d)	Diff.
		$''$	$''$	$''$	$''$	$''$	$''$	
$-20°$		$+3.35$	$+1.08$	-3.49	$+0.88$	-0.07	$+0.98$	$+1.05$
$-15°$	$0°$	$+2.38$	-0.30	-3.40	$+1.42$	-0.51	$+0.56$	$+1.07$
$0°$	$+15°$	$+2.21$	$+0.43$	-2.27	$+1.16$	-0.03	$+0.79$	$+0.82$
$15°$	$30°$	$+2.40$	$+0.60$	-2.39	$+0.37$	-0.00	$+0.48$	$+0.48$
$30°$	$45°$	$+2.34$	$+0.53$	-4.49	$+0.14$	-1.08	$+0.34$	$+1.42$
$45°$	$60°$	$+1.90$	$+0.96$	-1.55	-0.02	$+0.18$	$+0.47$	$+0.29$
Mean		$+2.43$	$+0.55$	-2.93	$+0.66$	-0.25	$+0.60$	$+0.85$

I have stopped at $+60°$ of declination because no result of value for our purpose could be derived beyond that limit. Were the stars affected only by a parallactic motion equal in the general mean in the case of different magnitudes the two columns before the last should (leaving out the effect of differences in the coefficients of the correction to the precession) be equal in the general mean. But there is evidently a difference of a systematic character, the mean proper motions near the meridians which include the solar apex being larger than those forming a right angle with that region. One way of accounting for this would be by supposing that, taking stars of equal apparent magnitude, those in region (a) which lies on the two sides of the vernal equinox are, in the general mean, farther away than those lying on the two sides of the meridian of twelve hours of right ascension. This amounts to the same thing as supposing that stars in the vernal region are somewhat larger in absolute magnitude than those in the autumnal region. The result would be that, comparing equal magnitudes, the stars (c) will have larger parallactic motions than the stars (a). The mere probable effect of the accidental inequalities of distribution and magnitude would result in a certain inequality of this kind, but the uniformity of the discrepancy prevents our attributing the difference to this cause alone.

We may more plausibly attribute the difference to possible systematic errors of BRADLEY's right ascensions, those near 0^h and 12^h of R. A. being too large relative to those near 6^h and 18^h. The amount of the correction which would annul the discrepancy is, approximately,

$$\varDelta\alpha = -0^s.033 \cos 2\alpha.$$

That so large an error as this could have affected BRADLEY's observations certainly seems unlikely. Yet, in view of its approach to constancy through $80°$ of declination, as shown by the column of differences, it seems difficult to attribute the discrepancy to any other cause. That systematic inequalities in the proper motions themselves could attain this magnitude and follow this law seems very unlikely.

It will, however, be of interest to examine more closely in what way the discrepancy is related to the right ascensions. I have therefore divided the sphere into eight lunes of right ascension, so situated that the central meridian on the initial lune should coincide with the assumed meridian of the solar apex, in eighteen hours and thirty minutes of right ascension. The normal equations for p formed from the numbers

in Table VII were added for each of these lunes through all seven zones, with the following results:

R. A.						(a)	(b)
					$''$	$''$	$''$
18.5;	$65.6\,\delta p$	$-\ 76.6X$	$-11.6Y$	$=+\ 77.3$	$\delta p =$	$+0.96$;	$+1.11$
21.5;	71.5	$-\ 53.2$	-68.8	$=+197.3$		$+0.47$;	$+0.58$
0.5;	74.4	$+\ 12.3$	-90.4	$=+224.2$		-0.08;	-0.10
3.5;	88.7	$+\ 78.4$	-60.8	$=+201.5$		$+0.38$;	$+0.25$
6.5;	100.5	$+108.9$	$+13.4$	$=+\ 34.1$		$+0.47$;	$+0.32$
9.5;	78.9	$+\ 53.7$	$+71.3$	$=-163.3$		$+0.07$;	-0.03
12.5;	59.8	$-\ \ 9.5$	$+71.7$	$=-214.7$		-0.54;	-0.51
15.5;	59.3	$-\ 55.9$	$+41.5$	$=-132.0$		-0.28;	-0.15
	598.7	$+\ 58.1$	-33.7	224.4			

Except on the meridian and antimeridian of the solar apex the several values of δp to be derived from these individual lunes will depend on the adopted value of the parallactic motions X and Y. If we use the values of X and Y as already derived from the conditional equations we should take the values for mag. 6.0, which is nearly the mean magnitude of the stars we have used. We shall then have

$$X = +0''.19$$
$$Y = -2.52$$

and, for $\delta\, p$, the results given above as (a). But if we take the solar apex as situate in 18 h. 30 m. of right ascension, we shall have

$$X = Y \cot. 277^\circ.5$$

and should therefore adopt

$$X = +0''.33$$
$$Y = -2.52$$

We shall then have the numbers given in column (b).

We may also combine the results in opposite hours of right ascension, so as to very nearly eliminate the effect of the parallactic motion from each pair of opposite lunes. This is done by multiplying either equation of a pair by such a factor that when added to the opposite equations, the coefficients of X and Y shall be so small that the uncertainty of the parallactic motion will not materially affect the result. Practically, however, it will suffice to take the mean result of each pair of opposite lunes. We thus have the following four values of δp:

R. A. h.	h.	$''$
6.5 and	18.5	$\delta p = +0.72$
9.5	21.5	$+0.27$
12.5	0.5	-0.31
15.5	3.5	$+0.05$

The dependence of the result upon the region of right ascension from which the elements of the problem are derived is brought out yet more strongly by this com-

parison. We have four pairs of lunes—one we may call the apical pair; that at right angles to it, the equinoctial pair; the other two, intermediate pairs. We have

from the apical pair		$\delta p = +0''.72$
"	" intermediate pairs	$+0.16$
"	" equinoctial pair	-0.31

If we take a mean of these three results, using the weights as derived from the normal equations, the weights will be nearly in the ratio 11, 20, and 9, and the result

$$\delta p = +0''.21,$$

substantially that already derived. But it is clear that since the result of the apical pair is free from all the uncertainty arising from the parallactic motion, it should receive a larger proportional weight. How much larger can not be determined by any mathematical process; I would suggest the factors 1.0, 0.7, and 0.5 as those which commend themselves to my judgment. The weights will then become 11,14 and 4.5, or 22, 28 and 9. The result for p will then be

$$\delta p = +0''.30$$

If we judge the probable error of this result by the discordances just shown, it would be at least $\pm 0''.25$. But if we consider the motion of the equinox as defined by the mean of the BRADLEY stars to be correct and the discordance to arise from systematic errors in the R. A.'s, then the discordance will cease to be an index to a probable error, and the simple mean $+0''.21$ will be the required result. But, as I have said, the inequality of parallactic motion may possibly be a factor in the result. I therefore consider the result $0''.30$ to be that given by the R. A.'s of the BRADLEY stars.

The question may arise whether the decisively larger value given by the apical pair of lunes is related to the parallactic motion, or grows out of the fact that a large proportion of the stars in these lunes is situated in the Milky Way. This question demands a special study of the galactic stars, with a view of determining whether they give results systematically different from those derived from other stars.

SECTION XIV.

Special examination of galactic stars.

It is well known that the condensation of stars in the region of the Milky Way begins with the lower magnitudes, and continually increases as we ascend to higher and higher magnitudes. The researches of KAPTEYN indicate that the galactic stars are, in the general average, farther than the others; in fact, that the galaxy may be considered as an annular collection situate outside the general boundary of the rest of the system. But it does not follow from this view that any large proportion of the galactic stars in the BRADLEY catalogue really belong to the galaxy as thus defined. Referring to KAPTEYN'S researches for a more exact statement of his conclusions, we shall here see what light may be thrown on the subject by an examination of AUWERS'S

Berlin A. G. catalogue for the zone $+15°$ to $+20°$. This catalogue is especially valuable for our purpose, owing to the very exhaustive discussion of the proper motions within its limits, which AUWERS has there given. Within this zone I take as the galactic regions,

	h. h.
Region A,	5.0 to 8.0;
Region B,	18.0 to 21.0.

These two regions include six hours of right ascension, or one-fourth of the entire surface of the zone. The corresponding numbers of stars contained in the catalogue are,

Stars in region A,	1,797
Stars in region B,	1,984
Total number galactic stars,	3,781
Outside of the galaxy,	6,008
Total number of stars,	9,789

It thus appears that 0.386 of the stars of the catalogue are contained with the galactic regions, the density being somewhat greater in region B.

The corresponding statistics for the number of stars which AUWERS has found to be affected by an appreciable proper motion are,

Proper motions in region A,	155
Proper motions in region B,	202
Total in the galactic regions,	357
Outside of the galaxy,	901
Total,	1,258

An equal distribution of the 1,258 proper motions would give 157 within each of the regions A and B. The actual number in region A falls short of this by 2, and that in region B exceeds it by 45.

On the other hand, were the numbers proportional to the stars, we should have,

In region A, 230 stars with proper motion;
In region B, 240 stars with proper motion.

It will be seen that there is no excess in the number of proper motions in region A. In the case of region B there is an excess which may lead us to suspect some condensation, yet the condensation is much less than proportional to the number of stars. The conclusion we may draw is that, assuming that the stars having sensible proper motions are all included within a sphere of nearly definite radius, the stars within this sphere do not show any well-marked condensation in the galactic regions, unless, perhaps, in region B near 19 h. of right ascension, and therefore do not belong to the galaxy proper, as defined by KAPTEYN.

the value of q' is larger for the higher magnitudes than for the lower ones. Although, when all the stars are considered, this parallactic motion is somewhat smaller than that already derived, the difference is not sufficient to show any marked diminution for the case of galactic stars.

We now pass to the more southern regions of the galaxy on each side of the sphere. The limits of these regions I have taken from Heiss's Atlas within each zone. By adding the normal equations in δp for each region, as given in Table VII, we have the following results:

Zone.	Limits of R. A.				″
30° to 45°	3 h.— 6 h.;	$15.2\delta p$	$+14.5X$	$-6.6Y$	$=+17.2$
	18 —22 ;	7.2	-11.3	-6.6	$=+15.2$
−15° to 30°	5 — 7 ;	16.0	$+15.8$	$+0.0$	$=+19.2$
	18 —21 ;	10.6	-13.4	-5.7	$=+13.8$
0° to 15°	5 — 8 ;	18.6	$+18.6$	$+1.9$	$=+7.7$
	17 —20 ;	13.4	-14.9	-2.2	$=+18.7$
−15° to 0°	6 — 8 ;	8.2	$+9.1$	$+2.3$	$=-7.7$
	17 —20 ;	16.4	-16.6	-3.0	$=+21.1$
−35° to −15°	7 — 8 ;	3.3	$+4.0$	$+1.7$	$=+0.4$
	17 —19 ;	14.0	-13.8	$+0.0$	$=+5.6$

The sum of these equations give the following normal equation in δp:

$$122.9\delta p = 111''.2 + 8.0X + 18.2Y$$

or

$$\delta p = 0''.905 + 0.065X + 0.148Y$$

Taking, as for mag. 6,

$$X = +0''.17$$
$$Y = -2.29$$

this will give

$$\delta p = +0''.58$$

while the value will be somewhat larger if we take the value of $q \cos D$ found from the stars of the galactic region alone.

If, to eliminate as far as possible the parallactic motion, we confine ourselves to the galactic stars south of 30° Dec., we shall have

$$100.5\delta p = 78''.8 + 11.2X + 5.0Y = 69''.3$$

and

$$\delta p = +0''.69$$

On the whole it does not seem that we are here dealing with anything else than the systematic changes of correction already found for the meridian and antimeridian of the solar apex. This periodic difference may be connected with the Milky Way or with the solar motion, it is impossible to say which; all that we can do at present is to refer it to positions in the heavens.

SECTION XV.

Discussion of the proper motions in declination.

The results of the preceding discussion of the right ascensions seem to indicate that we have to fear systematic errors of the observations and data more than irregularities of proper motion. Hence, in testing the declinations in the same way that we have the right ascensions, I first took zones 30° wide instead of 15°, and in each zone belts two hours wide in right ascension. I also varied a little the manner of taking the means within each trapezium; instead of an indiscriminate mean of all the stars from the fourth to the seventh magnitude, I assigned to the mean result for the stars of each magnitude a weight found by multiplying the number of stars by the following factors:

Mag.,	4	5	6	7
Factor,	0.2	0.3	0.4	0.5

The results of the process are shown in Table VIII. The four zones, whose limits are given at the top of the table, are designated by the letters A, B, C, and D. Here in each trapezium, after each of the four magnitudes, is given first the mean proper motion in declination of the stars of that magnitude, followed by the number in subscript figures of stars on which each mean depends. This is followed by the corresponding weight, and the weighted mean is taken as the final result within the trapezium.

In order to find the effect of the systematic corrections which I have applied to AUWERS's proper motions in declination, the same means were taken, using AUWERS's proper motions unchanged. The final results only of this process are given.

I have used these results, not to obtain a separate definitive value of the precession, but to judge of the probable magnitude of the systematic errors and the effect of the systematic corrections to AUWERS's proper motions. Owing to the combinations of all the proper motions within each region, it is not possible to make any distinction between the different values of ρ for the different magnitudes. Indeed from the way in which the stars are distributed it is probable that the effect of the different values of ρ will be but slight. Regarding the value of ρ as unity, each proper motion gives an equation of condition of the form

$$\cos\alpha\,(\varDelta n + X\sin\delta) + Y\sin\alpha\sin\delta - Z\cos\delta = \mu'.$$

In the combined equation we should take within each trapezium the mean values of the sines and cosines of the right ascension and declination which enter into the expression. I have, however, considered that it would answer our present purpose sufficiently well to regard these mean values as those corresponding approximately to the middle point of each trapezium. In fact, the general agreement among the numbers seems to show that the results would be changed very slightly by any probable deviation of these values. An exception may possibly be made in the case of the region within 30° of the pole.

Multiplying each of the twelve equations of the form last written by $\cos\alpha$ and adding the products, we shall have

$$6\,(\varDelta n + X\sin\delta) = \Sigma\,\mu'\cos\alpha.$$

TABLE VIII.

Zone.	0^h—2^h				2^h—4^h			
	° −27 to 0°	° 0 to +30°	° +30 to +60°	° +60 to +90°	° −30 to 0°	° 0 to +30°	° +30 to +60°	° +60 to 90°
	A	B	C	D	A	B	C	D
	″	″	″	″	″	″	″	″
Mag. 4	-3.7_{2} 0.4	-2.2_{12} 2	-1.4_{8} 2	$+1.1_{4}$ 1	$+1.3_{10}$ 2	-2.5_{13} 3	-1.6_{8} 2	-2.6 0.4
5	0.0_{9} 3	-2.5_{20} 6	-1.3_{18} 5	$+0.6_{8}$ 2	-0.4_{15} 4	-2.2_{25} 8	-2.2_{2} 6	$+1.2$ 1
6	-0.9_{16} 6	-0.8_{40} 16	0.0_{23} 9	0.0_{29} 12	-1.8_{13} 5	-2.4_{38} 15	-1.9_{16} 6	-0.1 5
7	$+1.2_{2}$ 1	-1.4_{10} 5	-1.6_{3} 2	-0.5_{5} 2	-4.8_{1} 1	-2.7_{14} 7	-1.6_{3} 2	$+0.5$ 3
Means	-0.55 10.4	-1.35 29	-0.69 18	$+0.08$ 17	-1.07 12	-2.36 33	-1.94 16	$+0.13$ 9.4

Zone.	4^h—6^h A	B	C	D	6^h—8^h A	B	C	D
Mag. 4	$+0.1_{10}$ 2	-2.4_{14} 3	-1.8_{6} 1	-0.9_{1} 0.2	-0.5_{15} 3	-3.0_{8} 2	-2.8_{2} 0.4	....
5	-1.5_{21} 6	-2.4_{44} 13	-1.6_{18} 5	-3.3_{2} 1	$+0.7_{19}$ 6	-3.3_{27} 8	-3.0_{12} 4	$+1.0_{5}$ 2
6	-0.4_{19} 8	-1.8_{51} 20	-2.3_{31} 12	-0.7_{8} 3	$+0.5_{14}$ 6	-1.6_{46} 18	-2.4_{25} 10	-1.2_{8} 3
7	-1.0_{3} 2	-2.7_{14} 7	-2.3_{7} 4			-1.1_{20} 10	-2.2_{5} 2	-2.7_{8} 4
Means	-0.78 18	-2.17 43	-2.12 22	-1.33 4.2	$+0.38$ 15	-1.90 38	-2.53 16.4	-1.38 9

Zone.	8^h—10^h A	B	C	D	10^h—12^h A	B	C	D
Mag. 4	-0.8_{3} 1	-3.7_{9} 2	-3.5_{5} 1	$+4.4_{2}$ 0.4	-3.1_{6} 1	-0.3_{8} 2	-7.8_{2} 0.4	-1.0_{1} 0.2
5	-1.0_{6} 2	-2.7_{20} 6	-2.6_{11} 3	-2.2_{7} 3	$+0.2_{13}$ 4	-2.4_{17} 5	-2.5_{10} 3	$+0.2_{2}$ 1
6	-1.3_{25} 10	-2.4_{38} 15	-1.6_{18} 7	-5.6_{4} 2	$+0.4_{15}$ 6	-1.4_{30} 12	-1.3_{23} 9	-2.5_{4} 2
7	-4.8_{2} 1	-0.9_{18} 9	-2.2_{5} 2	-4.9_{2} 1	$+1.1_{1}$ 1	-3.4_{9} 4	-4.0_{2} 1	-4.7_{1} 1
Means	-1.47 14	-2.12 32	-2.07 13	-3.27 6.4	$+0.10$ 12	-1.87 23	-1.96 13.4	-2.31 4.2

Zone.	12^h—14^h A	B	C	D	14^h—16^h A	B	C	D
Mag. 4	-5.1_{4} 1	-1.7_{4} 1	$+1.3_{1}$ 0.2	$+0.5_{3}$ 1	-3.5_{13} 3	-5.2_{9} 2	-8.3_{7} 1	$+1.0_{2}$ 0.4
5	-4.5_{9} 3	-2.1_{20} 6	-0.8_{14} 4.0	-2.4_{7} 2	-3.6_{19} 6	-1.7_{10} 3	$+1.5_{8}$ 2	$+1.2_{2}$ 1
6	-2.0_{26} 10	-3.0_{29} 12	-1.8_{12} 5	$+2.0_{2}$ 1	-3.6_{36} 14	-1.4_{20} 8	$+1.0_{4}$ 2	$+2.3_{1}$ 0.4
7	-1.5_{4} 2	-4.6_{4} 2	-2.0_{4} 2		-0.1_{1} 1	-3.2_{3} 2	....	....
Means	-2.60 16	-2.83 21	-1.42 11.2	-0.58 4	-3.44 24	-2.21 15	-0.66 5	$+1.69$ 1.8

Zone.	16^h—18^h A	B	C	D	18^h—20^h A	B	C	D
Mag. 4	-2.1 1	-0.8_{7} 1	$+2.1_{10}$ 2	$+0.1_{1}$ 0.2	-1.5_{8} 2	$+0.7_{12}$ 2	$+2.2_{12}$ 2	$+3.6_{2}$ 0.4
5	$\cdot 3.6_{14}$ 4	-2.0_{18} 5	$+0.7_{11}$ 3	$+0.8_{4}$ 1	-1.0_{25} 8	-0.1_{30} 9	$+0.9_{23}$ 7	$+2.3_{3}$ 1
6	$\cdot 3.4_{26}$ 10	-1.4_{30} 12	-0.4_{10} 4	$+0.6_{2}$ 1	-1.6_{36} 14	-1.1_{30} 12	-1.6_{10} 4	-1.3_{12} 5
7	2.9_{3} 2	0.0_{8} 4	-2.2_{2} 1	$+4.7_{1}$ 1	-3.2_{7} 4	-0.5_{9} 4	$+1.8_{4}$ 2	$+2.1_{2}$ 1
Means	-3.31 17	-1.25 22	$+0.24$ 10	$+1.97$ 3.2	-1.65 28	-0.54 27	$+0.53$ 15	-0.12 3.4

Zone.	20^h—22^h A	B	C	D	22^h—24^h A	B	C	D
Mag. 4	-1.3_{18} 4	-4.0_{9} 2	-1.0_{13} 3	-0.7_{2} 0.4	-1.0_{17} 3	-0.9_{7} 1	-0.7_{6} 1	-0.6_{2} 0.4
5	-0.8_{23} 7	-1.6_{33} 10	-0.5_{21} 6	$+1.4_{13}$ 4	-0.3_{18} 5	-1.1_{20} 6	-0.7_{20} 6	$+1.4_{13}$ 4
6	-0.7_{43} 18	-0.5_{23} 9	-0.4_{23} 9	$+1.8_{17}$ 7	-0.4_{36} 14	-1.3_{20} 8	-0.1_{22} 9	$+0.1_{17}$ 7
7	-2.2_{5} 2	0.0_{6} 3	$+1.6_{4}$ 2	-1.3_{5} 2	-1.1_{9} 4	-2.5_{2} 1	-0.2_{6} 3	$+0.2_{5}$ 2
Means	-0.90 31	-1.19 24	-0.32 20	$+1.14$ 13.4	-0.56 26	-1.28 16	-0.34 19	$+0.49$ 13.4

The results for the four zones are as follows. In deriving the second column of numbers from the first I have taken

$$X = +0''.20.$$

It must be noted that the $\Delta' n$ derived from AUWERS's equations applies to the STRUVE-PETERS's value of n.

Zone.	N's $\Delta n + X \sin \delta$	N's Δn	Wt.	AUWERS's $\Delta' n + X \sin D$	AUWERS's $\Delta' n$
	$''$	$''$		$''$	$''$
0° to − 36°	+0.59	+0.63	2	−0.13	−0.09
0° to + 30°	+0.45	+0.40	4	−0.54	−0.59
+ 30° to + 60°	+0.47	+0.34	3	−0.26	−0.39
+ 60° to + 90°	+0.81	+0.63	2	−0.40	−0.58
Mean	$\Delta n = +0.47$			$\Delta' n =$	−0.44
	or $\delta p = +1.17$			$\Delta n = +0.41$	
				$\delta p = +1.02$	

Applying these corrections to the respective values of n, we find the following mean results for the value of this quantity at the epoch 1850:

		$''$
From separate stars,	$100\, n =$	2005.32
" zones and regions,		2005.26
" AUWERS's prop. motions,		2005.20
L. STRUVE's result is		2004.55

This result may be subject to some doubt, owing to the breadth of the zones and the variation of the parallactic motion within that breadth. Still greater doubt may arise from the fact that the effect of the parallactic motion could not be rigorously taken account of in determining whether a star of considerable proper motion should or should not be included. Since the limits of inclusion are properly determined only by the magnitudes of the residuals after all corrections for precession and parallactic motion are made, it follows that a star which should be excluded when judged by the absolute amount of its proper motion, might be included when the residual proper motion became known and *vice versa.*

A yet narrower extension of the limits is suggested by the strength of the evidence, already presented, that nearly one-half the BRADLEY stars have no *motus peculiaris* in one coordinate exceeding 1''.0 per century. Could we with certainty pick out these stars from others, it would be well to depend upon them exclusively for the determination of the precession. But as we can not do so, an extension of the limits of proper motions to be included must be made, and we can then find no stopping place inside ± 7''. In order to use proper motions as near as possible to the final residuals in determining whether a star should or should not be included, and in order to test the whole process, I proceeded as follows with a new and nearly independent determination by zones 15° wide of declination and single hours of right ascension. I began by applying to AUWERS's proper motions in declination a

precessional correction corresponding to a diminution of STRUVE's 100 n by $0''.53$, and subtracting a parallactic motion

$$-2''.50 \sin\delta \sin\alpha + 0''.25 \sin\delta \cos\alpha - 1''.50 \cos\delta.$$

The sphere was divided into zones 15° wide north of the equator, while south of the equator only the single zone extending from 0° to -20° was used. Stars brighter than magnitude 4.0 were omitted, and all stars above this magnitude were included with equal weight except so far as exclusion on account of large proper motion was concerned, which, no distinction of magnitude being made, led to the exclusion of a larger proportion of the brighter stars.

As to the effect of the terms of parallactic motion, it is to be considered that the coefficient of $\cos\alpha$ is important, because it is in each zone united with the coefficient expressing the correction to the precession. Fortunately, the close approximation of the solar apex to eighteen hours of right ascension makes the term of parallactic motion depending on $\cos\alpha$ quite small; we may consider $0''.20$ and $0''.30$ as its probable limits. Moreover, it is diminished by being multiplied by $\sin\delta$, so that an error in its assumed magnitude diminishes to zero at the equator and changes its sign south of the equator. The term in $\sin\alpha$ is without effect, except so far as the stars may be unequally distributed near the two sides of the equinox. The adopted coefficient is so near that obtained for the mean of the included stars, and the sum total of the asymmetry on the two sides of the equinox is so small, that nothing would be gained by attempting to correct this motion from the equations in question. We shall, in fact, be nearer the truth by assuming the coefficient to be a known quantity. These considerations have led me to apply throughout each zone of declination the parallactic motion corresponding to the mean declination of the zone, as found by substituting the mean value of $\sin\delta$ and $\cos\delta$ for each zone in the above expression.

Within each zone the parallactic motion in declination will be merged with the mean systematic correction to AUWERS's proper motions within the zone. This correction is derived from an unpublished paper on the standard system of declinations.

The following are the expressions for the corrections actually applied to the proper motions of each zone, the last number including the systematic correction to the A. G. system as well $1''.5 \cos\delta$:

	° °	″	″	″
Zone A;	-20 to 0;	$+0.57 \cos\alpha$	$-0.40 \sin\alpha$	$+0.75$
B;	0 to $+15$;	$+0.50$ $+0.20$	$+0.32$	$+0.60$
C;	$+15$ to $+30$;	$+0.44$ $+0.04$	$+0.94$	$+0.50$
D;	$+30$ to $+45$;	$+0.38$ -0.42	$+1.50$	$+0.20$
E;	$+45$ to $+60$;	$+0.33$	$+1.97$	$+1.04$
F;	$+60$ to $+75$;	$+0.30$	$+2.30$	$+0.93$
G;	$+75$ to $+90$;	$+0.28$	$+2.45$	$+0.80$

I began by applying to the proper motions the additional terms $\Delta\delta_{n}$, derived from a comparison of the Boss standard stars with AUWERS-BRADLEY, and heretofore

considered as corrections applicable to the BRADLEY declinations. South of the equator this correction is zero. North of the equator the mean value within each zone is written under the coefficient of cos α up to 45°. For example, in zone B, the correction for precession and parallactic motion is $+0''.50 \cos\alpha$ and that for $\Delta\delta_\alpha$ is $+0''.20 \cos\alpha$. North of 45° $\Delta\delta_\alpha$ is omitted. After including it in the first three zones north of the equator I began to doubt its reality, and did not apply it in the remaining zones. There is thus a slight lack of similarity in the results actually used, but as the effect of the change in each zone is rigorously to change the result in the precessional motion by the amount of the coefficient, no confusion need result.

The proper motions having thus been arranged by zones, and those of each hour brought together, it became necessary to determine, in the case of each star, whether it should or should not be included on account of large proper motion. The adopted rule was to include all whose residual proper motion was now less than 7″, and to exclude all that exceed 10″. Between these limits a star was retained or omitted according to whether it had a small or a large proper motion in right ascension, and also according to the number of stars having motions of nearly the same amount. When, as frequently happened, two or three neighboring stars within the same zone had considerable proper motions of nearly the same magnitude, some would be struck out and others retained. In the case of the Pleiades the common proper motion of the group was considered as having the weight of two stars only. The whole process was conducted with great care to have no bias, which might be produced by any knowledge of the effect of an exclusion upon the final result.

Two other points in which the process differs from the previous one were that only stars within four hours of the equinox were included, it being considered that the weight of the precessional motion derived from stars within 30° of a solstice was too small to warrant their inclusion. Yet another point was that the proper motions of those stars whose declinations were not determined by BRADLEY were left out.

TABLE IX.—*Mean residual proper motions in declination for Bradley stars of magnitude 4.0 and upward, after subtracting precession* ($20''.0511 \cos\alpha$ *for 1850*) *and a parallactic motion of* $-2''.50 \sin\delta \sin\alpha + 0''.25 \cos\alpha - 1''.5 \cos\delta$.

	Zone A; −20° to 0°.			Zone B; 0° to +15°.			Zone C; +15° to +30°.			Zone D; +30° to +45°.		
R. A. h.	μ_0 ″	No. of stars.	Wt.	μ_0 ″	No. of stars.	Wt.	μ_0 ″	No. of stars.	Wt.	μ_0 ″	No. of stars.	Wt.
20.5	+0.63	32	11	+0.13	22	10	−0.90	19	9	−0.40	22	10
21.5	+1.12	30	11	+0.40	16	8	+0.11	14	8	−1.15	11	7
22.5	+0.64	36	12	+1.07	15	8	−0.15	11	7	−0.68	13	8
23.5	+0.40	28	11	+0.11	19	9	−0.97	18	9	+0.04	11	7
0.5	−0.77	17	9	+0.43	29	11	+0.21	23	10	−0.77	14	8
1.5	−0.06	17	9	−0.67	15	8	−0.62	27	11	−0.08	11	7
2.5	−1.28	19	9	−0.41	12	7	+0.15	27	11	−1.15	13	8
3.5	+0.54	19	9	+0.48	11	7	−1.84	20	10	−0.71	9	6
8.5	−0.60	16	8	+0.26	16	8	+0.23	33	11	+0.34	14	8
9.5	+0.20	25	10	+0.13	21	10	−0.84	17	9	+0.21	11	7
10.5	+1.20	19	9	+0.23	26	11	−0.06	20	10	+0.12	21	10
11.5	+1.60	10	7	−0.58	20	10	−0.41	13	8	−1.20	3	3
12.5	−0.81	15	8	−1.31	14	8	−0.95	29	11	+0.01	9	6
13.5	+0.30	28	11	−2.11	8	6	−1.63	11	7	+0.02	4	3
14.5	−1.84	21	10	−1.37	7	5	−1.01	10	7	+3.90	1	1
15.5	−1.00	26	11	−1.01	17	9	−1.51	16	8	−0.86	10	7

TABLE IX.—*Mean residual proper motions in declination for Bradley stars of magnitude 4.0 and upward, after subtracting precession ($20''.0511 \cos \alpha$ for 1850) and a parallactic motion of $-2''.50 \sin \delta \sin \alpha + 0''.25 \cos \alpha - 1''.5 \cos \delta$*—Continued.

	Zone E; +45° to + 60°.			Zone F; +60° to +75°.			Zone G; +75° to +90°.		
R. A. h.	μ_0 ″	No. of stars.	Wt.	μ_0 ″	No. of stars.	Wt.	μ_0 ″	No. of stars.	Wt.
20.5	−0.19	16	8	+0.24	13	7	−2.24	4	3
21.5	+0.27	11	7	−0.76	7	5	+0.03	5	4
22.5	−0.40	12	7	−0.14	8	6	+0.29	8	6
23.5	+0.66	18	9	−0.43	17	9	−0.40	4	3
0.5	+0.24	12	7	−0.12	12	7	−0.15	10	7
1.5	+0.72	12	7	+0.68	20	9	−0.24	6	5
2.5	+0.46	13	7	+1.32	12	7	+1.57	4	3
3.5	−1.28	12	7	+0.62	4	3	.....	..	..
8.5	+1.34	5	4	−0.81	9	6	.....	..	..
9.5	−0.11	8	6	+1.01	8	6	−1.77	2	2
10.5	+0.14	5	4	−2.97	3	3	+0.45	5	4
11.5	+0.58	4	3	+4.20	1	1	.....	..	..
12.5	−0.32	9	6	−0.85	6	5	+1.80	5	4
13.5	−0.87	11	7	−1.80	1	1	.....	..	..
14.5	−0.08	5	4	+1.65	2	2	+1.58	2	2
15.5	.....	..	..	−0.20	2	2	−0.89	2	2

TABLE X.—*Coefficients for solutions by least squares.*

Zone.	$[w]$	$[w \cos \alpha]$	$[w \cos^2 \alpha]$	$[w \mu_0]$	$[w \mu_0 \cos \alpha]$
				″	″
A	155	+ 6.5	+108.6	+ 3.9	+16.8
B	135	+ 1.1	96.3	−25.1	+45.8
C	146	+ 3.9	105.2	−93.8	+18.4
D	106	+13.8	73.1	−37.7	−37.3
E	93	+19.6	67.8	+ 3.2	+ 1.7
F	79	+24.4	56.6	+ 2.9	+12.3
G	45	+15.3	35.0	+ 3.4	− 9.2

The mean results within each zone and each hour of right ascension are shown in Table IX. It is then assumed that the mean of the proper motions corresponds to the central point of each trapezium, which is in the middle of each hour of R. A.

The second column of each block of the table contains the mean of the proper motions after the exclusions and corrections already described. This is followed by the number of included stars and the corresponding weight, the latter being deter mined on the hypothesis that the stars in each trapezium may have a proper motion equal to one-fourth the mean proper motion of the individual stars.

From each proper motion in each zone an equation of condition is formed between the correction to the precession in declination and the constant correction required to all the declinations within the zone. This equation is:

$$x + \cos \alpha \varDelta n = \mu_0$$

The reason for omitting the term in $\sin \alpha$ has already been given.

From the equations of each zone normal equations in x and $\varDelta n$ are formed. The coefficients of these equations are shown in Table X. The equations were solved

assumed that the mean of these motions should vanish when account is taken of the parallactic motion. It will be seen that within the zone 15°–30° there is strong evidence of a systematic error; this, however, must be discussed elsewhere.

SECTION XVI.

Comparison of the results of proper motions in declination with those of L. Struve.

The most remarkable feature of the results so far obtained is the discordance between the values of the precessional motion derived from the right ascensions and from the declinations. I have already remarked that this discrepancy extends through all the magnitudes, from the fourth upward. It is not materially diminished by any mode of treatment of either the right ascensions or the declinations. When a result from right ascensions is derived by using the LALANDE stars, the discrepancy is diminished. But I do not consider that the result from these stars has more than a small fraction of the weight to be attributed to the BRADLEY stars. We can not attribute the difference to the variation of personal equation with magnitude, because we have reason to suppose that this error is one which affects the modern more than the older observations, and which would, therefore, make the precessional motion derived from right ascensions come out too large. What adds to the embarrassment is that from substantially the same material—namely, a comparison of AUWERS-BRADLEY with modern observations—L. STRUVE derived a value of n so much smaller than mine that, accepting his result, the discordance would be reversed. In fact, the result of the direct solution of the equations derived from declinations, as found on page 18 of his paper * is,

$$\Delta n = -1''.09.$$

My provisional motion of n is less than the STRUVE-PETERS value used by L. STRUVE by 0''.85. Hence, L. STRUVE's result, considered as a correction to my provisional value, would be

$$\Delta n = -0''.24$$
$$\delta p = -0.60$$

This value of p diverges farther in the one direction than my own does in the other. The difference between our values of n is

$$\text{N.} - \text{L. S.} = +0''.68.$$

This enormous discrepancy seems to show that one result or the other must be seriously in error, either in the numbers used or in the mode of treatment. I have, therefore, sought to investigate its source, and applied every test I could think of to my own methods and conclusions.

The first question that would arise is whether the discrepancy extends through all the zones of declination, or is peculiar to some of them. I therefore made a rough solution of STRUVE's normal equations for Δn in his various zones, using his concluded

* Mémoires de L'Académie Impériale des Sciences de Saint-Pétersbourg. Septiéme série. Tome XXXV. Saint-Pétersbourg, 1887.

parallactic motion, and assigning equal weights to each of his trapezia. The assignment of equal weights prevents the numbers from being definitive ones; but the errors thus arising are of no importance for the purpose of the present comparison, which shows that the discrepancy extends through all the zones of declination, unless, possibly, those near the pole.

The stars of different magnitudes are very unequally weighted in the two investigations, STRUVE's diminishing much more rapidly with the brighter magnitudes than mine do. If the discrepancy arose from this cause, my results from stars of the higher magnitudes, say 6 and 7, should approach more nearly to those of STRUVE. But the actual divergence is in the opposite direction. The stars of magnitude $1-3$ give negative corrections in the preceding discussion. Hence, were a smaller weight given to these stars, my result would be increased, which would make it more divergent from the other.

I have applied the corrections $\Delta\delta_a$ to AUWERS's proper motions. These make the value of Δn very slightly larger. I was finally led to reject them for reasons mentioned in the preceding section, but doing so does not remove the discordance.

A very probable cause of the discrepancy would be the very different systems which we used in rejecting stars on account of large proper motion. STRUVE rejected only seven stars for this cause, and μ Cassiopeiæ is not included in the list of those rejected. It might therefore well happen that the stars which I have rejected, but which he included, would materially affect the result. With the view of determining whether this was the cause of the discordance, I have formed normal equations in the four unknown quantities from those stars which were omitted by me, but included by STRUVE. The results and solution are as follows:

Normal equations and solutions from large proper motions in declination used by L. Struve, but omitted in the preceding investigation.

Mag. 4.

$$
\begin{array}{rrrrcr}
90.1\delta p & -\ 2.3Y & +\ 60.6X & -\ 31.6Z & = & -1856'' \\
-\ 2.3 & +125.0 & -\ 3.5 & +\ 32.0 & & -1253 \\
+60.6 & -\ 3.5 & +127.3 & -\ 24.6 & & -1445 \\
+31.6 & +\ 32.0 & -\ 24.6 & +767.4 & & +8070
\end{array}
$$

Mag. 5.

$$
\begin{array}{rrrrcr}
188.9\delta p & +\ 6.1Y & +110.9X & -\ 38.2Z & = & +\ 1181'' \\
+\ 4.3 & +261.5 & -\ 7.7 & -\ 50.5 & & -\ 3177 \\
+106.5 & -\ 7.2 & +250.5 & +\ 26.0 & & +\ 175 \\
-\ 34.2 & -\ 48.8 & +\ 27.9 & +1662.0 & & +17758
\end{array}
$$

Mag. 6.

$$
\begin{array}{rrrrcr}
187.3\delta p & -\ 10.7Y & +121.6X & -\ 67.1Z & = & +\ 390 \\
-\ 10.8 & +217.4 & -\ 20.0 & -\ 80.4 & & -\ 4966 \\
+121.6 & -\ 20.0 & +283.7 & -\ 98.7 & & +\ 1906 \\
-\ 67.0 & -\ 80.4 & -\ 98.6 & +\ 1629.1 & & +11024
\end{array}
$$

Normal equations and solutions from large motions in declination used by L. Struve, but omitted in the preceding investigation—Continued.

Mag. 7.

$$\begin{array}{rrrrr} 38.3\delta p & -\ 3.4Y & +30.4X & -\ 39.6Z & =-\ 619'' \\ -\ 3.4 & +37.8 & -14.8 & -\ 56.4 & -\ 321 \\ +30.4 & -14.8 & +55.7 & -\ 21.8 & -\ 282 \\ -39.6 & -56.4 & -21.8 & +297.7 & +1951 \end{array}$$

Combined system.

$$\begin{array}{rrrrr} 591.3\delta p & -\ 13.3Y & +347.9X & -\ 199.4Z & =-\ 658'' \\ -\ 15.3 & +641.7 & -\ 46.0 & -\ 155.3 & -10587 \\ +343.5 & -\ 45.5 & +717.2 & -\ 119.1 & +\ 1024 \\ -195.3 & -153.6 & -117.1 & +4356.2 & +40174 \end{array}$$

Solution.

$$\begin{aligned} \delta p &= +\ 0.52'' \\ X &= +\ 1.72 \\ Y &= -\ 14.25 \\ Z &= +\ 8.78 \end{aligned}$$

There is, as we might anticipate, a good deal of discordance between the results from stars of different magnitudes. What is especially curious is the large negative absolute term given by stars of magnitude 4. This arises from a large number of stars of this magnitude around the vernal equinox having a negative proper motion. I regard this accumulation of negative proper motions as purely accidental.

It will be seen that the value of p derived from stars of large proper motion, while less than that from stars of smaller proper motion, is yet markedly larger than STRUVE's result. The combination of these stars with those of small proper motions, which I have used, would diminish the value of n by only a few hundredths of a second. It would seem that the discordance can not be accounted for in this way.

It would seem therefore that there must have been some discordance between the proper motions in declinations found by L. STRUVE from his comparison of AUWERS-BRADLEY with the PULKOWA declinations for 1855, and those derived by AUWERS from a comparison of the BRADLEY catalogue with a modern system for 1865. The method of reduction and comparison adopted by the two investigators, as described by them, was substantially identical. What we have therefore to do is to find whether there can be any systematic error in AUWERS's proper motions as found in his BRADLEY catalogue. The only error of the kind which would materially affect the result would be one depending on the cosine of the declination, or of the form

$$\Delta \mu' = k \cos \alpha.$$

The PULKOWA catalogue for 1865 is the best modern standard. I compared it with AUWERS's declinations for 1865, on which his proper motions rest, with a view of

detecting any error in the latter of the kind in question The discrepancy between AUWERS and PULKOWA for 1865 for each octant of right ascension is as follows:

Mean differences between the Pulkowa declinations for 1865 and those used by Auwers for the determination of proper motion.

Octant.	P.–A.	Octant.	P.–A.
	″		″
I	−0.26	V	−0.20
II	−0.28	VI	−0.20
III	−0.26	VII	−0.31
IV	−0.34	VIII	−0.28

It is clear that we have here no error of the kind sought for. If, therefore, there is any error of the kind sought for in AUWERS's proper motions, it must arise from some other cause than this.

In order to clear up the matter as thoroughly as possible, I made a comparison of AUWERS's proper motions with my own new standard still unpublished, which would be free from any hypothesis whatever as to the way in which AUWERS obtained his data. In this new standard catalogue there are about 800 stars whose proper motions are found in AUWERS'-BRADLEY, and were used in the preceding work. I made a comparison of these proper motions with those of AUWERS. The difference by octants is as follows:

Mean corrections of Auwers's proper motions as given by the standard of 1896.

Octant.	N.-A.	No. of stars.	Octant.	N.-A.	No. of stars.
	″			″	
I	−0.65	114	V	−0.20	96
II	−0.64	112	VI	−0.46	109
III	−0.70	98	VII	−0.53	125
IV	−0.78	86	VIII	−0.47	118

It is evident that we have no discordance depending on the right ascension which can explain the discrepancy. I am therefore obliged to regard the discrepancy in question as inexplicable, and to reach the conclusion that no other result than that reached in the present paper can be derived from the BRADLEY stars.

SECTION XVII.

Precessional motions derived from the R. A.'s of Lalande's zones.

The recent zone catalogues of the *Astronomische Gesellschaft* might throw much light on the question and guide us in adopting a mean value were it certain that the observations on which they are based were comparable with the ancient ones. In the case of the right ascensions there is little difficulty in securing this comparability in every point except one, the personal equation dependent on magnitude. The probable effect of this source of error can be best discussed after we have made some comparisons.

Of the older material the portion which seems best worth using is comprised in the zones of LALANDE. The later zones of BESSEL lose value not only on account of

their more recent epoch, but from the fact that they have not been completely compared with a modern system. In the case of two zones of the *Gesellschaft* the discussion of the comparisons with LALANDE is so complete that no difficulty will be found in studying the subject. We first remark that the LALANDE positions as now used rest on VON ASTEN's tables, which again depend on PIAZZI's positions for 1800. It is therefore to be presumed that the systematic corrections to reduce LALANDE's positions to a modern system are the same as in the case of the PIAZZI catalogue.

Comparisons of the older catalogues have been made in the first place with the A. G system. There is, however, a certain amount of doubt as to definition of this system as adopted by different investigators. Since the system in question is commonly supposed to be based on a comparison of the BRADLEY positions as determined by AUWERS for the epoch 1755, with the PULKOWA observations for 1865, it would naturally be supposed, and seems to have been very generally taken for granted, that the system in question is defined by this comparison. Such is not the case. By carrying back the A. G. positions to 1755 we do not reproduce the BRADLEY places, but a system of places in the general average about $0^s.035$ greater in R. A. This arises from the fact that AUWERS derived the proper motions by a comparison of BRADLEY with a system of modern positions for 1865 derived before the definitive positions were decided upon. When a decision was reached, it was found that the new system gave right ascensions about $0^s.035$ greater than those used in deriving the proper motions. But the latter were used without change. The result is the difference in question. AUWERS distinctly states this fact in the introduction to the fundamental catalogue in Publication XIV of the *Astronomische Gesellschaft*. But, nevertheless, the fact seems to have been very generally overlooked. The result of this state of things is that, in the absence of specific statement, we must be in doubt as to which interpretation has been employed. I shall, however, assume that AUWERS used the A. G. system and not the BRADLEY places.

Beginning now with AUWERS's Berlin zone, 15° to 20°, we find on pages 62–65 a very exhaustive comparison with PIAZZI. At the bottom of page 63 certain corrections to PIAZZI's right ascensions are found, which make them comparable with the A. G. system. The values vary but little with the right ascension, and the mean throughout the circle is

$$\Delta\,\alpha = +0^s.202.$$

This we take as the correction to reduce LALANDE's right ascensions to the A. G. system within the limits of the zone in question

Next, on pp. 230–246 of his catalogue, AUWERS gives a comparison of LALANDE's positions, brought forward to 1875 by precession alone, with the positions observed by him in his zone. I have taken the mean of the excess BERLIN-LALANDE for every hour of right ascension, omitting stars of which the deviation exceeded one second, and also those marked by AUWERS with an asterisk to indicate proper motion. The mean value of the excess is

$$\Delta\,\alpha = +0^s.175.$$

This agrees well with the mean of the comparisons given by AUWERS on page 67, where, however, the differences are given by zones and not by hours of right ascension.

The difference of the two corrections

$$\varDelta\ \alpha\ 1875 - \varDelta\ \alpha\ 1800 = -0^s.027$$

is to be taken as a correction to the precession during the interval between the observations. This differs slightly from the mean of the smoothed-off results given by AUWERS at the bottom of page 67, which would be about $-0^s.06$. But I shall use it for the correction to STRUVE's precession in 74 years. This gives a correction to STRUVE's 100m of $-0''.53$. But the centennial motion of the system N_1 is greater than that of the A. G. in R. A. by $1''.50$. Hence, for the system N_1 we should have as the correction to STRUVE's m,

$$100\ \varDelta\ m = +0''.97.$$

For the correction to the provisional values forming the basis of the present investigation, the result would be,

$$100\ \varDelta\ m = +2''.27.$$

This is an entirely inadmissible result, which must be attributed mainly to the error depending on personal equation in magnitude.

SECTION XVIII.

Comparison of Boss's Albany zone with Lalande.

In the Astronomical Journal, Volume IX, Boss discusses the constants of solar motion as derived from the comparison of his zone observations ($+1°$ to $+5°$) with the zones of LALANDE, BESSEL, etc. Incidentally he refers to the correction of the precessional constant which may thus be derived, reaching the conclusion that corrections of STRUVE's m and n, amounting to about $-0''.5$ are feebly indicated.

The comparisons with LALANDE which he publishes in the introduction to his zone, pages 22–23, lead to results admitting of more definite statement. He gives the following summary tables of comparison between the right ascensions of LALANDE, brought forward with precession alone to 1875, and those of his zone. The stars are divided into two classes, bright and faint, the first comprising the stars up to the seventh magnitude, the second those of higher magnitudes. The following are the results in each sextant of the circle of right ascension:

R. A.	Mag. 1 to 7 $\varDelta\alpha$	$\varDelta\delta$	Mag. 7.8 to 10. $\varDelta\alpha$	$\varDelta\delta$	Bright - Faint. $\varDelta\alpha$	$\varDelta\delta$
	s	"	s	"	s	"
0 h. to 4 h.	+0.11	−3.6	+0.14	−3.8	−0.04	+0.0
4 to 8	+0.14	−3.7	+0.16	−3.3	−0.05	−0.2
8 to 12	−0.08	−2.7	+0.03	−2.2	−0.10	−0.4
12 to 16	−0.03	−3.7	+0.06	−2.8	−0.09	−0.1
16 to 20	+0.10	−2.7	+0.13	−3 1	−0.03	−0.3
20 to 24	+0.17	−2.5	+0.20	−4.0	−0.02	+0.9
Means	+0.07	−3.2	+0.12	−3.2	−0.05	0.0

In this table the Albany right ascensions are not corrected for personal equation depending on magnitude. When thus corrected the mean $\Delta\alpha$ for the group 1 to 7 would have been

$$+0^s.07 - 0^s.03 = +0^s.04;$$

and the mean $\Delta\alpha$ for the group 7.8 to 10 would have been

$$+0^s.12 - 0^s.07 = +0^s.05.$$

The mean of these two is $+0^s.045$.

Next, for the reduction of the LALANDE positions to the modern system, Boss formed a "special catalogue," comprising the few fundamental stars of the A. G. system which lay within his zone, and a large number based on the combination of BRADLEY'S positions with the modern A. G. positions, mistakenly supposing that he thus has a system comparable throughout with the A. G. The comparison of the LALANDE catalogue with this special catalogue is as follows:

	R. A.		Decl.			R. A.		Decl.	
α	Obs.	$\Delta\alpha$	Obs.	$\Delta\delta$	α	Obs.	$\Delta\alpha$	Obs.	$\Delta\delta$
		s		″			s		″
0 *h.*	27	+0.15	22	−3.2	12 *h.*	26	+0.11	27	−2.7
2	21	+0.10	18	−2.0	14	11	+0.17	11	−2.4
4	14	+0.16	14	−2.7	16	23	+0.20	20	−1.5
6	22	+0.21	18	−2.1	18	13	+0.16	14	−2.6
8	8	+0.01	8	−2.4	20	10	+0.19	10	−3.9
10	16	+0.19	16	−2.3	22	21	+0.11	21	−2.3

The general mean value of $\Delta\alpha$ is $+0^s.147$. Subtracting this from $0^s.045$, the modern comparison when the Albany positions are corrected for personal equation, we have a difference $-0^s.102$ as a correction to the general precession in right ascension during a period of eighty-four years. This gives as the correction to the centennial precession in right ascension of STRUVE and N_0, respectively, the amounts,

$$\Delta\, 100\, m = -\overset{s}{0.122} = -\overset{\prime\prime}{1.82} \text{ (corr. to STRUVE)}$$
$$\Delta\, 100\, m = -0.034 = -0.52 \text{ (corr. to } N_0\text{)}.$$

But the general mean motion in right ascension of the system N_1 is greater than that of the BRADLEY-A. G. system by $0^s.070$. We have, therefore, relatively to the system N_1 the correction of the general precession N_0 in right ascension

$$\Delta\, 100\, m = 1''.05 - 0''.52 = +0''.53.$$

We may accept this as the definitive correction given by the comparison of Boss and LALANDE.

I have made no attempt at an independent investigation of the comparisons in declination, because of the unknown possible systematic error of the LALANDE-PIAZZI declinations. It may be remarked, however, that the correction of $-0''.5$ given by Boss to STRUVE'S precession in declination is eouivalent to a correction of the provisonal value of $100n$, amounting to

$$\Delta\, 100\, n = +\overset{\prime\prime}{0.34}.$$
$$\Delta\, 100\, m = +0.80.$$

The interesting feature of these results is that they bring the correction of m more nearly into accordance with n. But in accepting the results from the right ascension as real, we meet the difficult question of the influence of magnitude on personal equation. My investigations on this subject, which will appear with some fullness in the next part of the *Astronomical Papers*, and of which a summary is given in the *Astronomical Journal*, Volume XVI, p. 65, lead to the conclusion that in the case of the leading principal observatories this personal equation has nearly the same value for eye and ear observations that it has for those registered on the chronograph. Although this result was unexpected, and is probably contrary to the general impression, we have a very simple explanation, nearly identical with that now accepted as the cause of the corresponding error in the case of the chronographic observations. In both cases the observer is biased in his estimate of the distance of the star from the transit thread by the apparent magnitude of the star. When the latter is apparently smaller, his unit of measurement is unconsciously taken less in proportion, and the distance from the wire is therefore assumed to be greater. In the case of eye and ear observations this erroneous estimate before the star crosses the thread, should, in the general mean, be neutralized by the corresponding error on the opposite side after the star has crossed the thread. Were the position of the star at the first clock-beat following transit estimated by the observer with the same leisurely precision as at the moment of the clock-beat next preceding, it is to be supposed that the error would not exist. But, in the case of modern observations, when the observer only has fifteen or twenty seconds between the several threads in which to record his observation and prepare for the transit over the thread next following, he is naturally hurried and fails to make a sufficiently careful estimate after the star has passed the thread. His result is therefore affected by the bias of the approaching star.

In the case of the older transits, the observations were either made over a single thread, or the threads were so wide apart that the observer had plenty of time for his estimate. It is therefore quite likely that in these cases the error in question does not exist. A comparison of Boss-LALANDE seems to show that it did not exist in LALANDE's observations. We must not, however, too hastily conclude that by the simple device of observing a star over a single thread, or by taking plenty of time for the estimate, the error would necessarily be avoided. There might be fear that, in any case, the estimate made after the star had crossed the thread would be biased by the preceding expectation of the observer. This bias ought, however, to be done away with through proper discipline on the observer's part.

Returning to the two preceding values, since the error in question was absolutely determined by Boss in the case of his zone and is therefore eliminated, we may regard the result of his comparison with PIAZZI as one not affected by this error. In the case of SCHJELLERUP the presumption would be in favor of the existence of the error. The result seems to show, however, that it was very small. But we can not, on this ground alone, assume that it vanished, and therefore assign a large weight to his result.

SECTION XX.

Elimination of the parallactic motion from the precession of each individual star.

There is yet a fourth mode of treating the subject, which is eminently worthy of consideration, although I am unable to apply it exhaustively. If we regard the position of the solar apex as known, then we may eliminate the parallactic motion from the two equations given by each particular star, and thus obtain an equation in which the precession appears as the only unknown quantity. This amounts to the same thing as determining the magnitude of the precession by the condition that the apparent rotation of the whole system of stars around the direction of the solar apex as an axis shall vanish. The apparent motion of precession may be considered as one taking place around the pole of the ecliptic as an axis. It is therefore a fortunate circumstance in the application of this method that the distance between the solar apex and the pole of the ecliptic is only about 30°. The result is that our equations do not lose much in weight by the elimination in question. The latter is conducted in the following way. Let us put

A, D; the R. A. and Dec. of the solar apex,

q; the parallactic motion of a star.

We then have for the components of the solar motion,

$$X = \rho q \cos D \cos A$$
$$Y = \rho q \cos D \sin A$$
$$Z = \rho q \sin D$$

We also put

$$a' = \cos D \sin (\alpha - A)$$
$$b' = \cos D \sin \delta \cos (\alpha - A) - \sin D \cos \delta$$

By substitution of these values of X, Y, and Z, in the equations of p. 33 they become

$$a\delta p + a'q = \mu \cos \delta$$
$$b\delta p + b'q = \mu'$$

Eliminating q from these equations and putting for brevity

$$\Pi = a'b - ab'$$

we have

$$\Pi \delta p = a'\mu' - b'\mu \cos \delta$$

By substituting in Π the values of a, a', b, and b', we see that it may be thrown into the form

$$A \sin^2 \delta + B \sin \delta \cos \delta + C \cos^2 \delta + E$$

and this again into the form

$$L + M \sin 2\delta + N \cos 2\delta$$

the coefficients L, M, and N being functions of α. Thus, a', b', and Π may be tabulated as functions of α and δ.*

*These formulæ are not the best for the purpose, since they ignore the fact that near the solar apex the parallactic motion vanishes. As the result derived in this section does not enter directly into the conclusion, I deem it unnecessary to study the best formulæ at present.

Our problem requires the determination of the values of A and D, which detemine the position of the solar apex. The determination of this quantity which we have already made should not be regarded as the definitive one, because it depends solely on stars with small proper motions. If the stars had no absolute proper motion, it is clear enough that the solar motion could best be determined from the stars having the largest parallactic motion, because they would be the nearer. In any case, in the general mean, the presumption is that the stars with a larger proper motion are the nearer to us and therefore best adapted to the determination in question. At the same time the method which has always been pursued of selecting stars having large proper motions and determining the parallactic motion from them alone is not free from objection. The determination of the proper motion of a star presupposes a knowledge of the magnitude of the precession; and a star may or may not be included in the list of those having a sufficient proper motion, according to whether one value or another of the precession is used.

It seems to me that the numerous results obtained by past investigators should now give way to the three latest determinations by L. STRUVE, BOSS, and STUMPE. From the BRADLEY stars L. STRUVE found:

$$A = 273^\circ.4; \quad D = 27^\circ.3.$$

But, owing to the systematic correction to BRADLEY's declinations which I have found, I estimate that his parallactic motion Z requires a correction of $+0''.70$ to reduce it to the standard which I have called C_{96}. With this correction we may put the results into the form

$$\begin{aligned} X &= +0''.23 \\ Y &= -3.87 \\ Z &= +2.70 \end{aligned}$$

This gives, as L. STRUVE's corrected result,

$$\begin{aligned} A &= 273^\circ.4 \\ D &= 34.9 \end{aligned}$$

BOSS's general result (A. J., Volume IX, page 165) is

$$\begin{aligned} A &= 280^\circ \\ D &= 40 \end{aligned}$$

STUMPE, in his very exhaustive discussion (Astronomische Nachrichten, volume 140, pages 177–190), gives a number of separate results which have the great advantage of being derived from southern stars as well as northern stars. I have made a general combination of them to derive what seems to me the most probable final result from his work, and find it to be,

$$\begin{aligned} A &= 279^\circ.5 \\ D &= 38.7 \end{aligned}$$

Altogether it seems to me that we may take the most likely position of the solar apex as,

$$A = 277^\circ.5 = 18^h\ 30^m$$
$$D = +38.0$$

This conclusion places the solar apex in the middle of the eighteenth hour of right ascension, in a direction different from that of the star α Lyræ by a quantity not exceeding its probable error.

After preparing tables based on this adopted position of the solar apex and on the preceding formulæ, I find that these formulæ are not those best adapted to the work, and that they coincide with the best formulæ only near what we may call the sun-way equator, or the belt 90° distant from the solar apex. In all other positions, the measures being considered as those of rotation around the solar apex, it will follow that each star enters into the result with a weight proportional to the cosine of the sun-way latitude, or to the sine of its distance from the apex. At the apex itself the rotation indicated by the proper motion necessarily enters the equation with a vanishing weight, and therefore gives no result whatever. But it is evident that, since the parallactic motion vanishes at the solar apex, all stars near this apex may be included in the result without the necessity for eliminating the parallactic motion. It is true that since such stars are only about 30° distant from the pole of the ecliptic they will, in any case, only enter with a small weight, yet they should, in theory, be included. The method in question is substantially equivalent to the determination of the precession from the proper motion of the star in sun-way longitude only. I find that this method of determination was proposed several years ago by Kapteyn, but I became aware of it too late to utilize the idea in the form in which he presents it. It differs from my form only in the practical introduction of a different system of weights for stars distant from the sun-way equator.

My method of proceeding was determined by the following considerations. The parallactic motion is not only eliminated from the precession in the immediate neighborhood of the apex, where it vanishes, but it is eliminated from the proper motion in right ascension on both the meridian and the antimeridian of the apex. That is to say, from proper motions in right ascension of stars near these meridians we have a determination of the precession from which the solar motion is eliminated.

I consider that all stars within one hour of the apical meridian and antimeridian satisfy this condition, for, although at a distance of 15° from that meridian, the effect of the parallactic motion will be quite appreciable; it would be in opposite directions on the opposite sides of the meridian, and therefore ought to be eliminated from the mean result. Taking out the two lunes thus defined from the sphere, we have left two regions, the one extending in R. A. from $19^h.5$ through the vernal equinox to $5^h.5$; the other from $7^h.5$ through the autumnal equinox to $17^h.5$. I shall call these two regions the vernal and autumnal regions, respectively. Within them I took all the stars which were within 30° of the sun-way equator or, more exactly, all those for which the value of the coefficient Π exceeded 0.80, and from the proper motions in each region a value of the precession was derived by the formulæ which should be independent of the parallactic motion.

In the case of the apical lunes I simply took the normal equations in right ascension as already formed for all the individual stars, except those of the first and second magnitudes. A combined equation for each lune was then formed from the separate ones by the factors for magnitudes already given with the following results:

$$\text{Apical lune;}\quad 1558\,\delta p - 1621X - 258Y = +1826''$$
$$\text{Antiapical lune;}\quad 2940\,\delta p + 2674X + 216Y = +2367$$

From each of these combined equations we may obtain a result in two ways. In one way we assume that the parallactic motion is eliminated from the mean, and simply ignore X and Y. In the other way we substitute the values of X and Y as already found for the standard of the fifth magnitude stars. Beginning with the apical lune, when we omit X and Y, we have

$$\delta p = 1826 : 1558 = +1''.17.$$

Retaining X and Y, the absolute term becomes

$$1826 + 535 - 650 = 1711,$$

whence

$$\delta p = +1''.10.$$

Passing to the antiapical meridian we have, when we omit X and Y,

$$\delta p = +0''.81.$$

Retaining them, the second member of the equation becomes

$$2367 - 543 + 544 = 2368,$$

whence we have the same value, $+0''.81$.

The discordance of the results on the apical and the antiapical lunes would be removed by taking a smaller right ascension of the solar apex, say, the value already found in this investigation. But I do not consider that this affords any sound evidence for doubting the superiority of the adopted value. Table XI shows the computation of the motions in the vernal and autumnal regions, bounded by the apical lunes.

TABLE XI.—*Computation of the precession from 962 stars near the ecliptic and the sun-way equator by eliminating the parallactic motion from the equations given by each star.*

Vernal region; $19^h.5$ to $5^h.5$ R. A.							Autumnal region; $7^h.5$ to $17^h.5$ R. A.						
Mag.	Stars.	$\Sigma r \sin \Delta$	$\Sigma \Pi$	Prod. by F^2.		δp.	Mag.	Stars.	$\Sigma r \sin \Delta$	$\Sigma \Pi$	Prod. by F^2.		δp
		″		″		″			″		″		″
3	18	+15.4	15.5	5.5	5.5	+0.99	3	19	−13.7	16.1	−4.9	5.8	−0.85
4	64	46.0	55.4	29.4	35.5	+0.83	4	44	+0.1	37.7	+0.1	24.1	0.00
5	149	56.5	128.3	56.5	128.3	+0.44	5	120	+24.0	100.1	+24.0	100.1	+0.24
6	231	189.3	195.7	272.6	281.8	+0.97	5	214	+3.9	182.2	+5.6	58.3	+0.02
7	56	56.2	48.6	110.2	95.3	+1.16	7	47	−3.4	40.5	−6.7	79.4	−0.08
	518			474.2	546.4	+0.87		444			18.3	267.7	+0.07

It will be seen that the stars have been first classified according to the magnitudes, as in the former portions of the work. In the third column the symbol τ may be considered as representing the proper motion of the star on a great circle resolved in the direction of sun-way longitude. $\varDelta$ represents the distance of the star from the solar apex. The combination $\tau \sin \varDelta$ is then the product of the proper motion by the appropriate factor when a solution is made by least squares in order to represent $\tau \sin \varDelta$ by the precessional motion.

The next column, $\Sigma\Pi$, contains the sum of the corresponding coefficients of the lunisolar precession when normal equations for this precession are formed.

In order to form the final normal equation these partial normals are multiplied by the squares of the factor already found, which are taken to represent the relative mean distances of the stars of the several magnitudes. The factors then are, respectively,

0.6, 0.8, 1.0, 1.2, and 1.4.

There is an evident systematic difference in the results from the two regions, the largest result from the autumnal region being smaller than the smallest result from the vernal region. This difference extends through all the magnitudes, and can therefore not be considered as accidental.

For the sake of study we place the four results just derived from stars in corresponding but not identical regions, in juxtaposition with those already derived from the right ascensions alone. In the case of the latter I have taken the results already given for the octants including the apex and the antiapex, and the weighted means of the intermediate octants:

	From R. A.	From motion in sun-way longitude.	
	$''$	$''$	
Apical region . . .	$\delta p = +1.04$	$+1.14$;	wt. = 1
Vernal region . . .	$+0.30$	$+0.87$;	2
Antiapical region . .	$+0.40$	$+0.81$;	1
Autumnal region . .	-0.21	$+0.07$;	2

Giving double weight to the results from the vernal and autumnal regions, the number of stars being nearly double, the result of eliminating the parallactic motion is

$$\delta p = +0''.64.$$

As to the R. A's I have already indicated that the discordances, or the small values given by the intermediate octants, may be accounted for by supposing a systematic or accidental difference of absolute magnitudes in opposite quarters of the sphere. Were such the case, any such difference should be eliminated with the solar motion, and therefore not appear in the results derived after that motion is eliminated. It will be seen that the discordance is removed in the case of the vernal regions, but not in that of the autumnal regions. It is also remarked that, supposing this difference of magnitude to be real, it affords an explanation of the larger value found for the declinations. A very little consideration will show that if the stars of a given apparent magnitude are farther away within the vernal region than within the autumnal region, then the smaller parallactic motions in the former region will tend to diminish the precession

found from the right ascensions, and increase that found from the declinations. The same effect will be produced by the relatively greater negative parallactic motions of the nearer stars in the autumal region.

But in view of the almost uniform extension of the discordance through all the apparent magnitudes, as shown by a comparison of the colums δp in Table XI, I still think the main cause of discordance to be an error of double period in the BRADLEY R. A.'s.

SECTION XXI.

Summary and discussion of results.

The various results for the precessional motion derived and discussed in the preceding pages rest on two independent bases, the observed motions in right ascension, and those in declination. The motion in sun-way longitude does not rest on a third independent basis, but should rather be considered as a particular combination of the other two motions. What we have finally to do is to decide independently upon the most probable result of the motions in right ascension alone, then on the most probable result from the declinations alone, and then upon the best combination of the two.

Taking first the right ascensions, we have found from the BRADLEY stars, by different modes of treatment, the following values for correction to the value N_0 of the lunisolar precession:

I.	From individual stars, each with equal weight	$\delta p = +0''.24$
II.	From groups of stars, each having an adjusted weight;	$+0.20$
III.	By division into lunes, each an octant in extent, giving greater weight to the lunes nearer the meridian of the solar apex;	$+0\ 30$
IV.	By the statistical method;	$+0.37$

Of these four results I conceive that the third is best entitled to acceptance as the definitive value to be derived from the motion of the BRADLEY stars in right ascension. While the results of the three other methods are not to be left wholly out of consideration we can not say that their combined result would be decidedly different from $0''.30$, some being greater and some less.

From the LALANDE stars we have found:

By comparison with Boss's Albany zone;	$+0''.57$
By comparison with SCHJELLERUP;	$+0\ 67$
Mean, ...	$+0.62$

Of the two results here combined I conceive that there can be no doubt of the legitimacy of including Boss's result, since the injurious effect of variation of personal equation with magnitude was carefully determined and eliminated from the comparison of his right ascensions with those of LALANDE. The one weak point is that his zone was only 4° wide. Against this is to be placed the consideration that the large number of stars observed by LALANDE within this zone were most carefully reduced by comparison with a sufficient number of standard stars in the zone itself. On the other hand we have no evidence that SCHJELLERUP'S right ascensions are free from this

Were it possible to assign any law of the planetary precession for the past and future in this way, this proceeding might be worthy of consideration. But in view of the necessary uncertainty which attaches to the homogeneity of the system N_1 itself, and of the impossibility of forming any hypothesis of the past and future value of the planetary precession otherwise than on the basis of theory; considering also the undoubted character of the theory, and the fact that no admissible changes in the masses of the planets can change the amount of this motion by more than one or two tenths of a second per century, I conceive that this mode of proceeding is inadmissible. We should rather anticipate that the astronomy of the not distant future will reconcile the two motions with theory.

The second method is to consider $\Delta E'$ as an unknown correction of which the most probable value is zero, but which has a probable error, to be determined as best we can. This probable error should be combined with that of the absolute term $0''.36$. The question would then merge itself into that of the probable error to be assigned to the value of p as derived from the right ascensions alone. Independently of $\Delta E'$ we have found that the values of p derived from pairs of lunes of opposite right ascensions, from the mean of which the parallactic motion is eliminated, differ by more than $1''$. This would seem to imply a probable error of not less than $0''.30$ in the absolute term of p. But if we regard the point of reference from which p is to be measured as the actual mean of all the stars around the circle of right ascension, this discrepancy becomes simply a periodic term in the correction of the proper motions, and no longer indicates a probable error of the absolute term.

Passing now to $\Delta E'$, under the hypothesis at present under consideration its probable error would be simply its probable value, which, in the absence of any evidence as to that value, we might estimate at $\pm 0''.20$. On the whole we may consider that the range of judgment as to what probable error should be assigned to the value of p as derived from the right ascensions is quite large, and that it might be placed anywhere between $\pm 0''.20$ and $\pm 0''.40$.

In the case of the value derived from the declinations there are fewer sources of uncertainty. It may be placed between $\pm 0''.30$ and $\pm 0''.40$. Considering the combination, there seems to be a possible range of doubt whether we should regard the result from the declinations as entitled to more weight than the right ascensions, or as entitled only to one-third the weight. Altogether it would seem that, from this point of view, we can only say that, according to different judgments, the value of δp might range between limits $+0''.50$ and $+0''.80$, or

$$+0''.50 < \delta p < +0''.80.$$

The third method of procedure is to anticipate the probable correction which must hereafter be applied to the motion of the equinox, or the probable value of $\Delta E'$. Although a definitive determination of this correction is not possible at the present moment, there are data for an approximate estimate. The Greenwich observations of the Sun during the sixty years since 1834 are of great value for this purpose, owing to their having been made on a uniform system by so large a number of observers that the personal equation peculiar to the limb of the Sun is probably nearly elimi-

nated from the final result. I find that taking the Sun's absolute longitude as given in my new theory, a rough investigation of the Greenwich results during the sixty years, 1835–1895, leads to the approximate value,

$$\varDelta E' = +0''.5.$$

I also conceive that the observed right ascensions of Mercury are worthy of consideration in this respect, because they are nearly free from the effect of the personal error affecting the limbs of the Sun. It is true that the Conference of 1896 expressed a preference for determining the equinox from the Sun alone, owing to the manner in which uncertain elements of Mercury would enter into a result derived from observations of that planet. I conceive that this objection is of force only as regards the determination of the Sun's absolute longitude from the declination of Mercury. When observations of right ascension alone are considered, the reduction of an observed right ascension of Mercury to the center of its motion is one all the errors of which will be nearly eliminated in any one revolution of Mercury in its relative orbit. I conceive therefore that the correction $\varDelta E' = +1''.0$ given by observations of Mercury alone is one legitimately worthy of consideration.

These corrections are not to be accepted by themselves alone, but are to be considered in connection with the great mass of material on which the equinox for the system N_1 is based. This material of course gives the value zero for the correction in question. Altogether I conceive that the existing material justifies us in anticipating for this correction a value

$$\varDelta E' = +0''\ 30.$$

Accepting this result, we shall have from the right ascensions alone,

$$\delta p = 0''.36 + 1.09\ \varDelta E' = +0''.69.$$

The combination with the declinations will then lead to a probable result somewhere between the limits

$$0''.80 < \delta p < 0''.90.$$

We have thus a correction $+0''.80$ which appears to be about the upper limit of probability if we regard the system N_1 as that to which our equinox is to be referred, and the lower limit in the event of the correction which now seems most probable having to be applied in the future.

In what precedes we have considered the question of the relative weights to be assigned to the results from the right ascensions and declinations as the result of a somewhat uncertain judgment. There is, however, one principle which will lead us to a fairly definite combination, although I have not investigated it rigorously. I have already remarked that a possible cause for the discrepancy between the two results is to be found in an inequality of the average magnitude of stars in the autumnal and vernal regions of the heavens, and a consequent inequality in the parallactic motion of stars of equal apparent magnitude in the two directions. This cause will produce opposite effects on the precessional motions as derived from right ascension and from declination. The amount of the effect does not admit of absolute statement, because it would depend upon the distribution of the inequality in different

parts of the heavens. Now, the elimination of the parallactic motion from the equations given by each star eliminates this error completely, and at the same time affords a criterian for the relative weights of the R. A.'s and the Dec.'s. We have found in this way

$$\delta p = +0''.64$$

as that of the combination in question.

A correction of $+0''.30$ to all the motions in right ascension would increase this result by a quantity which I have not accurately determined, but which would not differ much from $0''.20$. With this increment we should therefore have, by eliminating the parallactic motion—

$$\delta p = +0''.84.$$

The question now arises, whether it is advisable to anticipate this correction to the motion of the equinox. It seems to me that it is, for these reasons: It is to be expected that the system of right ascensions will be corrected from time to time to bring it into accordance with the observed equinox. Such corrections will be productive of little trouble or confusion. But a change in the precessional motion will be productive of confusion to astronomers, while a small error in its amount will not cause any trouble. It may, therefore, be anticipated that such value of the precessional motion as may now be generally adopted will be continued through a large part of the twentieth century, perhaps the whole of it, should no large correction be found necessary.

From all these considerations I have considered that the correction

$$\delta p = +0''.80$$

would be very near the best compromise to adopt. This is very nearly 0.00016 of the lunisolar precession itself. I have, therefore, increased the provisional value of the precessional constant by this fraction of its whole amount, and reconstructed the precessional motions accordingly. I find that, owing to the defects of decimals of the computations employed in "Elements and Constants," the actual correction thus derived to the lunisolar precession there given is

$$\delta p = +0''.82.$$

I therefore propose this correction as the value which, considering all the data for judging the future, seems most likely to meet the requirements of the case. If it be too small, the injurious effect on n will apparently only be about $0''.12$ per century, even if we assign to the latter the value derived from the declinations alone. If it prove to be too large, the excess will probably not produce any confusion in astronomy, since it will, in any case, be but a small fraction of the parallactic motion. We therefore have as the definitive conclusion of the preceding investigation the following value of the precessional constant as I have defined it and of the precessional motions for 1850:

Precessional constant;	$P - 5490''.66 - 0''.00364T$
General precession;	$p = 5024.53$
Lunisolar precession;	$p = 5036.84$
	$100\,m = 4607.11$
	$100\,n = 2005.11$

With this value of the precessional constant I have gone over the investigation of the motion of the equator found on pages 196–202 of my "Elements and Constants," with some variations of methods, designed to make the whole more rigorous and homogeneous. The results are given in the next section.

SECTION XXII.

Numerical values of the precessional motions.

The following tables show the numbers to be used in reductions of the mean places of the fixed stars from one epoch to another when we use the preceding value of the precessional constant, and the secular motion of the ecliptic as deduced from the masses of the planets employed in my tables of the Sun. The data are of two classes, those which express the annual motions of precession, and those serving for the rigorous trigonometric reduction of star places from one epoch to another.

Values of the centennial precessional motions from 1725 to 2000.

	General preces.	Lunisolar precess.	100 m.	100 m_s.	100 n.	log. 100 n.	100 n_s.	log. 100 n_s.
	″	″	″	s	″		s	
1725	5021.75	5036.22	4603.62	306.908	2006.18	3.302369	133.745	2.126278
1750	5022.30	5036.34	4604.32	306.955	2005.96	3.302323	133.731	2.126232
1775	5022.86	5036.47	4605.01	307.001	2005.75	3.302277	133.717	2.126186
1800	5023.41	5036.59	4605.71	307 048	2005.54	3.302231	133.703	2.126140
1825	5023.97	5036.71	4606.41	307.094	2005.32	3.302185	133.689	·2.126094
1850	5024.53	5036.84	4607.11	307.141	2005.11	3.302139	133.674	2.126048
1875	5025.08	5036.96	4607.80	307.187	2004.90	3.302092	133.660	2.126001
1900	5025.64	5037.08	4608.50	307.234	2004.68	3.302046	133.646	2.125955
1925	5026.19	5037.21	4609.20	307.280	2004.47	3.302000	133.632	2.125909
1950	5026.75	5037 33	4609.90	307.327	2004.26	3.301954	133.617	2.125863
1975	5027.31	5037.45	4610.60	307.373	2004.04	3.301908	133.603	2.125817
2000	5027.86	5037.58	4611.29	307.420	2003.83	3.301862	133.589	2.125771

Quantities expressing the relative positions of the equator and equinox at any two epochs, for use in the trigonometric reduction of mean places of the fixed stars from one epoch to another.

We consider the quadrangle formed by the poles of the equator and ecliptic at two different epochs. One of these epochs we call the zero epoch. It is supposed to be that for which the position of a star is given and to which the various quantities are in the first place referred. The other epoch is considered to be a variable one. We represent the respective poles of the ecliptic and equator by the symbols E and P, those which refer to the zero epoch being called E_0 and P_0.

We join the points E_0 and P so as to divide the quadrangle into two spherical

triangles E_0 P_0 P and E_0 P E. We represent the parts of these triangles which enter into the theory as follows:

$\varepsilon_0 = E_0\, P_0$, or the obliquity of the zero equator to the zero ecliptic.
$\varepsilon_1 = E_0$ P, or the obliquity of the actual equator to the zero ecliptic.
$\varepsilon = E$ P, or the obliquity of the actual equator to the actual ecliptic.
$\theta = P\, P_0$, the distance between the poles, or the obliquity of the actual to the zero equator.

$\psi =$ Angle $P_0\, E_0$ P, or the lunisolar precession upon the zero ecliptic.
$90° - \zeta_0 =$ Angle $E_0\, P_0$ P.
$90° - \zeta =$ Angle E_0 P P_0.
$\lambda =$ Angle E_0 P E, the planetary precession on the equator, or the arc of the actual equator intercepted between the two ecliptics.

$z = \zeta - \lambda$.

The formulæ embodying the application of the preceding quantities to the reduction of the fixed stars differ from those of BESSEL, used by CHAUVENET, in that the epoch for which the poles E_0 and P_0 is supposed to be given is the same for which the position of the star is given. In the other theory the zero epoch is a constant for the whole theory, so that three different positions of each pole enter into the theory. This change involves a slight change of notation, which seems advisable in order to avoid confusion. The notation here adopted differs from that in the concluding pages of "Elements and Constants" only in that I have here put ζ_0 for what is there called ζ_1.

The reduction in question involves three constants, θ, ζ_0 and z, of which the use is as follows. We put,

α_0, δ_0, the coordinates of a star as given for an epoch t_0 which I call the zero epoch.
α, δ, the coordinates as reduced to an epoch t by precession only.

To find α and δ we compute,

$$a_0 = \alpha_0 + \zeta_0$$

$$p = \sin\theta\,(\tan\delta_0 + \tan\tfrac{1}{2}\theta\cos a_0)$$

$$\tan(a - a_0) = \frac{p\sin a_0}{1 - p\cos a_0}$$

$$\tan\tfrac{1}{2}(\delta - \delta_0) = \frac{\cos\frac{1}{2}(a + a_0)}{\cos\frac{1}{2}(a - a_0)}\tan\tfrac{1}{2}\theta$$

$$\alpha = a + z$$

As defined by these equations, the constants θ, ζ_0, and z are functions of t_0 and t, and can be most conveniently expressed as functions of t_0 and $t - t_0$, because then they vary very slowly with t_0. Their values for any pair of epochs between 1600 and

2100 may be found from the following expressions, which are given for three values of the zero epoch, t_0, and where we put

$$\tau = \frac{t - t_0}{250}$$

Zero epoch.				
	° ′ ″	″	″	″
1600;	$\varepsilon = 33\ 29\ 28.69$	$+0.5513\tau^2$	$-0.1206\tau^3$	$-0.0002\tau^4$
1850;	$23\ 27\ 31.68$	$+0.4078$	-0.1207	-0.0002
2100;	$23\ 25\ 34.56$	$+0.2645$	-0.1206	-0.0002
	″	″	″	
1600;	$\psi = 12589.01\tau$	$-6.66\tau^2$	$-0.02\tau^3$	
1850;	12592.06	-6.69	-0.02	
2100;	12595.15	-6.71	-0.02	
	″	″	″	
1600;	$\zeta + \zeta_0 = 11545.65\tau$	$-6.12\tau^2$	$+0.54\tau^3$	
1850;	11551.30	-6.14	$+0.54$	
2100;	11556.98	-6.16	$+0.54$	
	″	″	″	
1600;	$\zeta - \zeta_0 = 45.30\tau$	$-9.91\tau^2$	$-0.03\tau^3$	
1850;	33.54	-9.91	-0.03	
2100;	21.77	-9.92	-0.03	
	″	″	″	
1600;	$\lambda = 45.30\tau$	$-14.83\tau^2$	$-0.05\tau^3$	
1850;	33.54	-14.86	-0.05	
2100;	21.77	-14.88	-0.05	
	″	″	″	
1600;	$\zeta_0 = 5750.17\tau$	$+1.90\tau^2$	$+0.28\tau^3$	
1850;	5758.88	$+1.89$	$+0.28$	
2100;	5767.61	$+1.88$	$+0.28$	
	″	″	″	
1600;	$z = 5750.17\tau$	$+6.82\tau^2$	$+0.30\tau^3$	
1850;	5758.88	$+6.83$	$+0.30$	
2100;	5767.61	$+6.84$	$+0.30$	
	″	″	″	
1600;	$\theta = 5018.10\tau$	$-2.66\tau^2$	$-0.64\tau^3$	
1850;	5012.77	-2.66	-0.65	
2100;	5007.44	-2.67	-0.65	

As an example of the use of these expressions, let us suppose the epoch for which α and δ are given to be 1875. By interpolation of coefficients to $t_0 = 1875$, we find the following expression for ζ_0, z, and θ

$$1875;\quad \zeta_0 = 5759.75\tau + 1.89\tau^2 + 0.28\tau^3$$
$$z = 5759.75\tau + 6.83\tau^2 + 0.30\tau^3$$
$$\theta = 5012.24\tau - 2.66\tau^2 - 0.65\tau^3$$

If we take $t = 1900$ as the epoch to which α_0 and δ_0 are to be reduced, we should put

$$\tau = 0.10,$$

which would give,

$$\begin{aligned} \zeta_0 &= 575''.993 \\ z &= 576.043 \\ \theta &= 501.196 \end{aligned}$$

as the constants for reduction from 1875 to 1900.

A check on these values is afforded by computing the values for reduction from 1900 to 1875. For $t_0 = 1900$ we find the expressions

$$\begin{aligned} \zeta_0 &= 5760''.62\,\tau + 1''.89\,\tau^2 + 0''.28\,\tau^3 \\ z &= 5760.62\,\tau + 6.83\,\tau^2 + 0.30\,\tau^3 \\ \theta &= 5011.70\,\tau - 2.66\,\tau^2 - 0.65\,\tau^3 \end{aligned}$$

For $t = 1875$ we then have $\tau = -0.10$, and

$$\begin{aligned} \zeta_0 &= -576''.043 \\ z &= -575.994 \\ \theta &= -501.196 \end{aligned}$$

If we put the values of ζ_0, z, and θ last found in parentheses we should have,

$$\begin{aligned} (\zeta_0) + z &= 0 \\ (z) + \zeta_0 &= 0 \\ (\theta) + \theta &= 0 \end{aligned}$$

It will always be advisable to test the values of these quantities in this way. The same test may be applied to the expressions tabulated on p. 75, by taking the epoch t as 1600, 1850, and 2100 in succession, and combining each with the other two zero epochs. We then have $\tau = \pm 1$ or ± 2, as the case may be. This process controls pretty completely the entire numerical work by which the values of ζ_0, z, and θ on p. 75 were derived.